<u>Welcome to Lecture 4!</u> Linear Transformations

<u>This week</u>: Wed HW due
Office Hours, 12-2pm
891 Evans

Fri Quiz through §1.9

"When images become inadequate,
I shall be content with silence."
Ansel Adams

Warmup Problem Consider list of vectors in $\mathbb{R}^5$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v}_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Find the <u>smallest</u> k such that $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ is linearly dependent.

$$\underline{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(Such k must exist since 6 vectors > 5 dim space)

<u>Soln</u> Place vectors in cols of matrix

$$\begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_6 \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Seek soln $(x_1, \ldots, x_6)$

where $x_k \neq 0$

but $x_{k+1} = \cdots = x_6 = 0$.

Want smallest such soln

Strategy: put in REF and take k the index of the first <u>free</u> col (no pivot)

$$\rightsquigarrow \text{REF} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{2} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

first <u>free</u> col $\boxed{\text{Take } k=5}$

Set $x_5 = 1, x_6 = 0$

Solve for $x_1, \dots, x_4$ $\dots \left(-1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 1, 0\right)$

We find :

$$-1\,\underline{v}_1 + \frac{1}{2}\underline{v}_2 + \frac{1}{2}\underline{v}_3 + \left(-\frac{1}{2}\right)\underline{v}_4 + \underline{v}_5 = \underline{0}$$

so $\underline{v}_1, \dots, \underline{v}_5$ <u>lin dep</u>.

Note also :

$\underline{v}_1, \dots, \underline{v}_4$ <u>lin indep</u>

since pivot in cols $1, \dots, 4$.

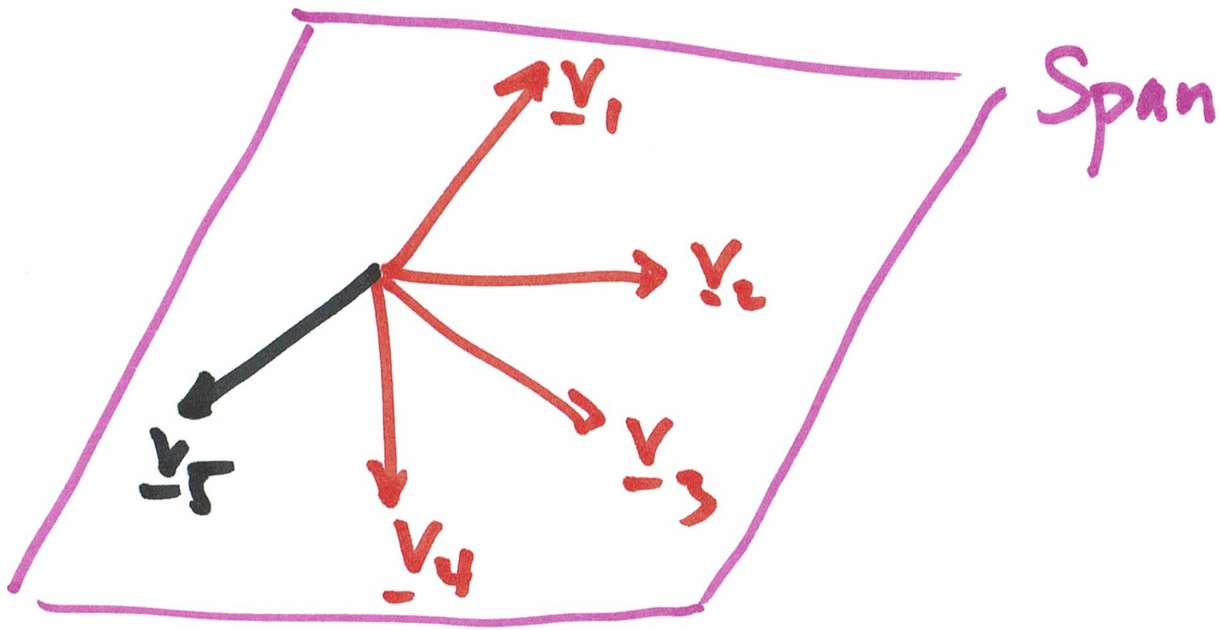
We can rewrite lin dep eqn in form:

$$\underline{v}_5 = \underline{v}_1 - \frac{1}{2}\underline{v}_2 - \frac{1}{2}\underline{v}_3 + \frac{1}{2}\underline{v}_4$$

<u>Interpretation</u>:

$$\underline{v}_5 \in \mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_4\}$$

Cartoon:



Thm $\underline{v}_1, \dots, \underline{v}_n \in \mathbb{R}^m$ lin dep

$$\Updownarrow$$

there is some k such that

$$\underline{v}_k \in \text{Span}\{\underline{v}_1, \dots, \underline{v}_{k-1}\}$$

Idea of Pf Place vectors as cols of matrix.
Take k = index of (first) free col.

Recall diff perspectives:

1) Lin Systs

2) Aug. matrices $[A \mid \underline{b}]$

3) Vector eqns $x_1 \cdot \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n = \underline{b}$

Now one more:

4) Lin transformations $A\underline{x} = \underline{b}$

where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Let's think of the coeff matrix

$$A = \begin{bmatrix} | & & | \\ \underline{a}_1 & \cdots & \underline{a}_n \\ | & & | \end{bmatrix} \text{ as a } \underline{\text{map}}$$

$$\underset{\text{input, domain}}{\mathbb{R}^n} \xrightarrow{\ A\ } \underset{\text{output, target}}{\mathbb{R}^m}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto A\underline{x} = x_1\underline{a}_1 + \cdots + x_n\underline{a}_n$$

Think of coeff matrix A as a map taking n-vectors to m-vectors

Interpretation of solving $A\underline{x} = \underline{b}$

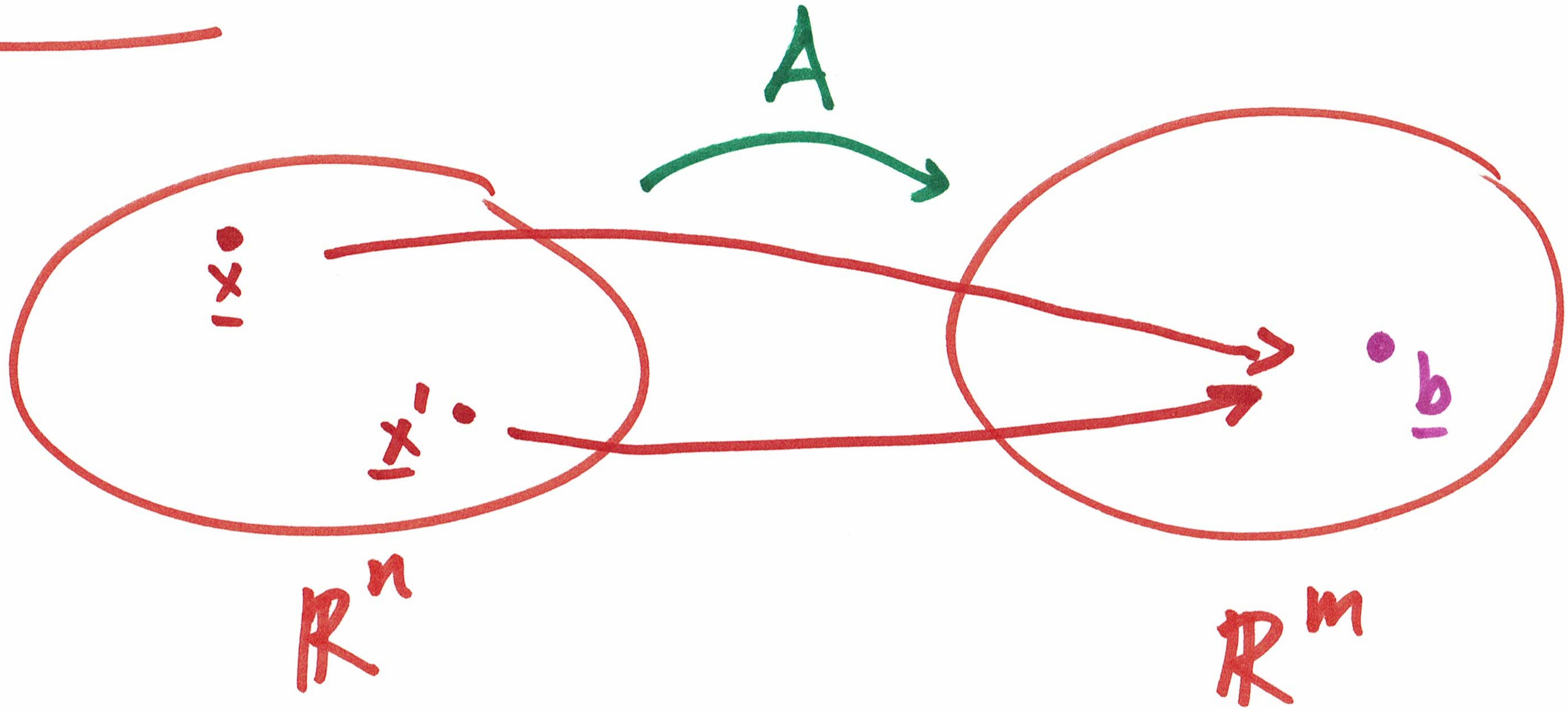
<u>Existence</u>?

Is there $\underline{x} \in \mathbb{R}^n$ so that A takes $\underline{x}$ to $\underline{b} \in \mathbb{R}^m$?

<u>Uniqueness</u>?

How many $\underline{x} \in \mathbb{R}^n$ does A take to $\underline{b} \in \mathbb{R}^m$?

Cartoon

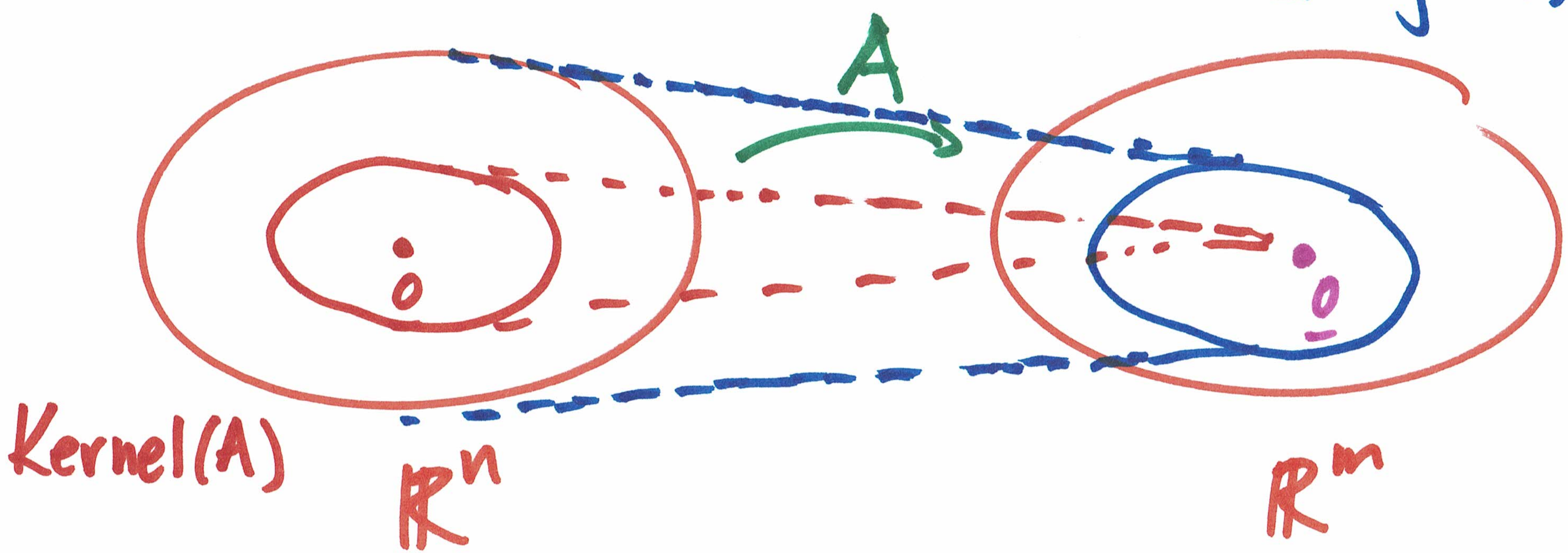


<u>Def</u> Given $A = \begin{bmatrix} | & & | \\ \underline{a}_1 & \cdots & \underline{a}_n \\ | & & | \end{bmatrix}$ thought of as a map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

1) $\text{Image}(A) = \text{Range}(A) = \{ \underline{y} \in \mathbb{R}^m$ s.t. there is $\underline{x} \in \mathbb{R}^n$ with $A\underline{x} = \underline{y} \}$

2) $\text{Kernel}(A) = \text{NullSpace}(A) = \{ \underline{x} \in \mathbb{R}^n$ s.t. $A\underline{x} = \underline{0} \}$

Cartoon



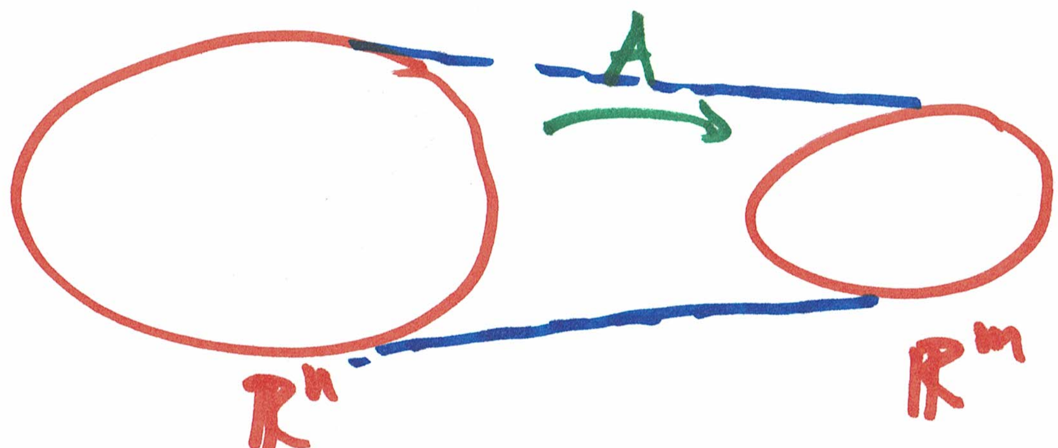
Observe

1) Image $(A) = \{ \underline{b} \in \mathbb{R}^m$ so that $[A \,\vdots\, \underline{b}]$ has soln $\}$

2) Kernel $(A) = \{ \underline{x} \in \mathbb{R}^n$ that solve $[A \,\vdots\, \underline{0}] \}$

__Def__ Given $A = \begin{bmatrix} | & & | \\ \underline{a}_1 & \cdots & \underline{a}_n \\ | & & | \end{bmatrix}$ thought of as a map $\mathbb{R}^n \to \mathbb{R}^m$

1) A is <u>onto / surjective</u> if $\text{Image}(A) = \mathbb{R}^m$



2) A is <u>one-to-one / injective</u> if whenever

$$A\underline{x} = A\underline{x}' \text{ we have } \underline{x} = \underline{x}'$$

Lemma A is one-to-one $\iff$ $\text{Kernel}(A) = \{\underline{0}\}$

Pf ($\Rightarrow$) Suppose A is one-to-one

Take any $\underline{x} \in \text{Kernel}(A)$. This means

$A\underline{x} = \underline{0}$. But $A\underline{0} = \underline{0}$ So $\underline{x} = \underline{0}$.

($\Leftarrow$) Suppose $\text{Kernel}(A) = \{\underline{0}\}$

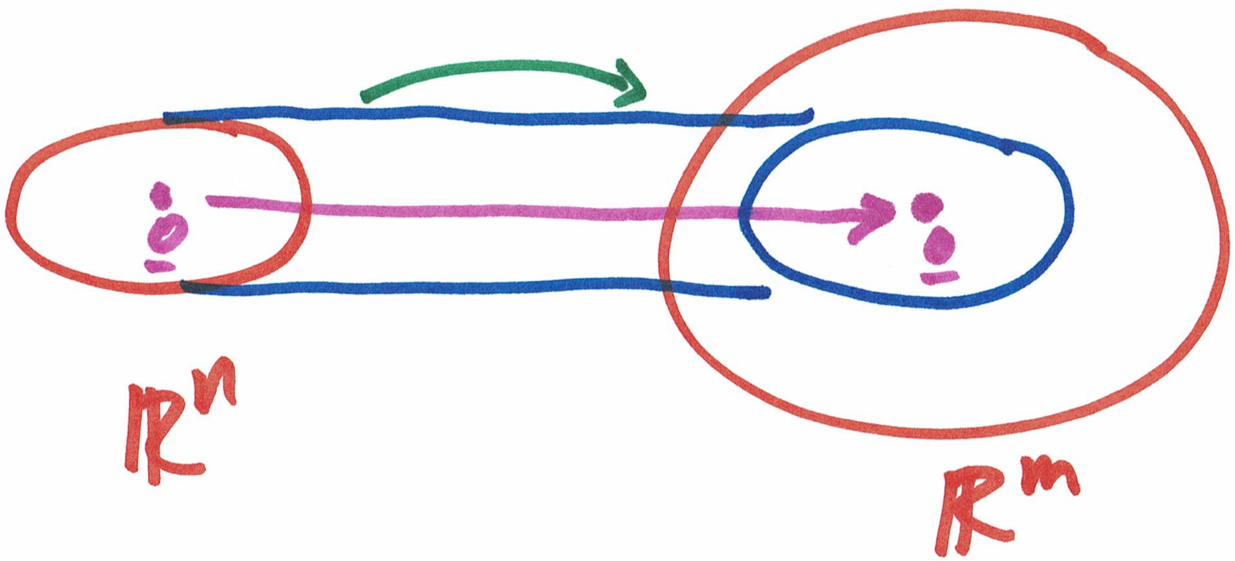
Take some $\underline{x}, \underline{x}' \in \mathbb{R}^n$ so that $A\underline{x} = A\underline{x}'$

Consider $\underbrace{A\underline{x} - A\underline{x}'}_{=\,\underline{0}} = A(\underline{x} - \underline{x}')$

Check this!

This means $\underline{x} - \underline{x}' \in \text{Kernel}(A)$

So $\underline{x} - \underline{x}' = \underline{0}$ so $\underline{x} = \underline{x}'$ $\blacksquare$



__Observe__

1) A onto $\iff$ for all $\underline{b} \in \mathbb{R}^m$ we can solve $[A \,\vdots\, \underline{b}]$

$\iff$ cols of A span $\mathbb{R}^m$

$\iff$ pivot in each row of A

2) A one-to-one $\iff$ $\underline{0}$ is only soln to $[A \,\vdots\, \underline{0}]$

$\iff$ cols of A lin indep

$\iff$ pivot in each col of A

Ex For what c is $A = \begin{bmatrix} 1 & 0 & 1 \\ c & 1 & 0 \\ 0 & c & c \end{bmatrix}$ onto? one-to-one?

Soln **Cases**

$c=0$ $A = \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ neither onto nor one-to-one

$c \neq 0$ $\rightsquigarrow$ REF $\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -c \\ 0 & 0 & c-c^2 \end{bmatrix}$

Cases

- **$c=1$** neither
- **$c \neq 0,1$** both onto and one-to-one

<u>Thm</u> <u>Suppose $m=n$.</u>

1) A onto $\Longleftrightarrow$ one-to-one.

2) $\cancel{A}$ $\text{Image}(A) = \mathbb{R}^m \Longleftrightarrow \text{Kernel}(A) = \{0\}$

3) $[A \,\vdots\, \underline{b}]$ has soln for all $\underline{b} \in \mathbb{R}^m$ $\Longleftrightarrow$ $[A \,\vdots\, \underline{0}]$ has unique soln $\underline{0} \in \mathbb{R}^n$

What is special about map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

given by matrix A?

<u>Two properties</u>:

1) $A(\underline{x}+\underline{x}') = A\underline{x} + A\underline{x}'$

2) $A(c\,\underline{x}) = c\cdot(A\underline{x})$

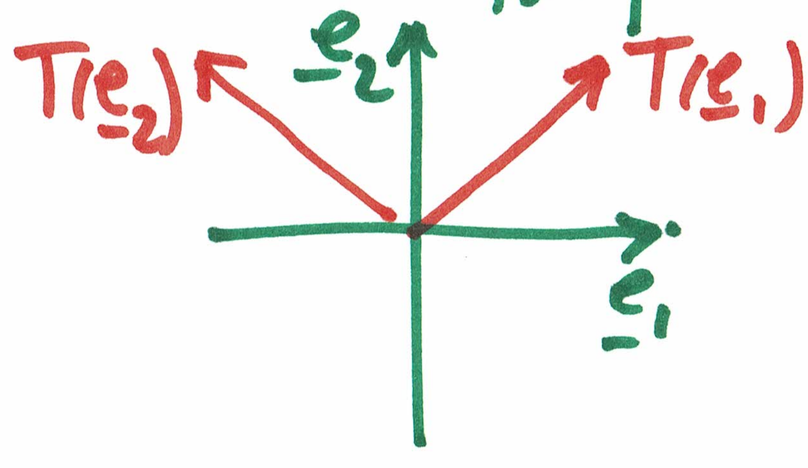
Def A lin transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is any map satisfying above two properties.

Amazing fact: Any lin transf $T: \mathbb{R}^n \to \mathbb{R}^m$ is given by some $m \times n$ matrix A!

$$A = \begin{bmatrix} | & & | \\ T(\underline{e}_1) & \cdots & T(\underline{e}_n) \\ | & & | \end{bmatrix} \qquad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Exer Find matrix A of lin transf $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates plane 45° counterclockwise.

Soln $A = \begin{bmatrix} | & | \\ T(\underline{e}_1) & T(\underline{e}_2) \\ | & | \end{bmatrix}$ $\qquad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$