Welcome Back! Lecture 25 Fourier Series!

Or Why I Became a Mathematician

This week: Wed Office Hrs 12-2pm, 891 Evans

Fri Last Quiz! through § 10.4

Next week: Reviews during lecture meetings

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$ Wow!

"The deep study of nature is the most fruitful source of mathematical discoveries."

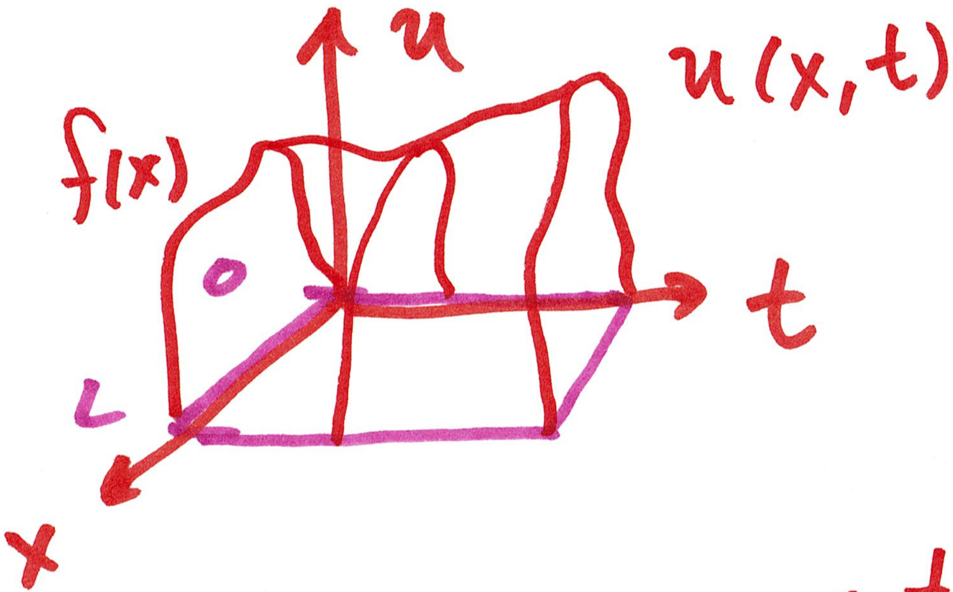
Joseph Fourier

__Goal__ (Both our goal and Fourier's original goal)

To solve Heat Eqn $\partial_t u(x,t) = \beta \partial_x^2 u(x,t)$

t = time, x = location in rod of length L

$u(x,t)$ = temp. at x at time t



time = 0

IVP: $u(x, \overset{0}{\cancel{t}}) = f(x)$

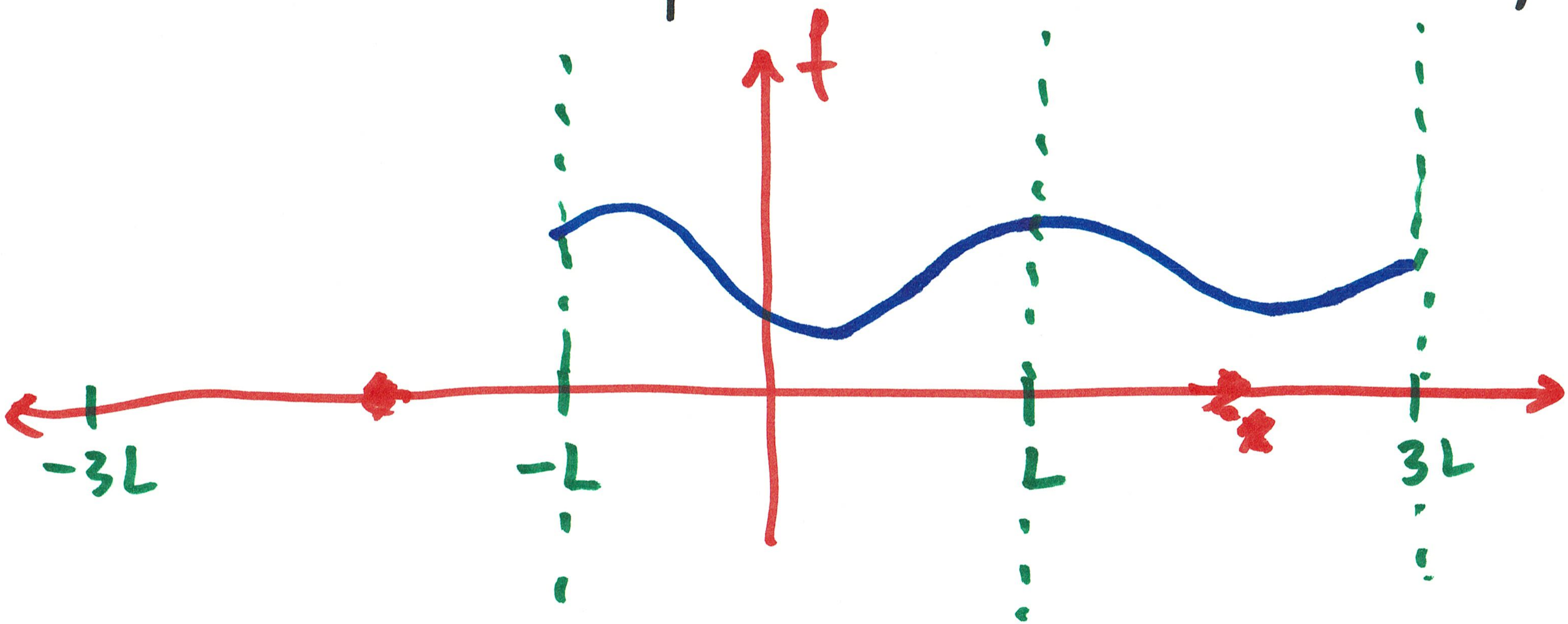
BVP: $u(0, t) = 0 = u(L, t)$

ends of rod

Next time solve using Fourier series!

Fourier series: study of good "bases" and their "coords" in vect sp

$$V_L = \{ f : \mathbb{R} \to \mathbb{R} \text{ periodic of period } 2L \quad f(x+2L) = f(x) \}$$



__Alternative__ __formulation__

$$V_L = \{ f : [-L, L] \to \mathbb{R} \text{ with } f(-L) = f(L) \}$$

__Rmk__ We'll often take $L = \pi$

So $V_L = \{ 2\pi\text{-periodic fns } f : \mathbb{R} \to \mathbb{R} \}$

and we think of x as an __angle__.

<u>Inner</u> <u>product</u> on V_L

$$\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)\,dx$$

<u>Ex</u> Calc $\|f\|$ for $f(x) = 1$.

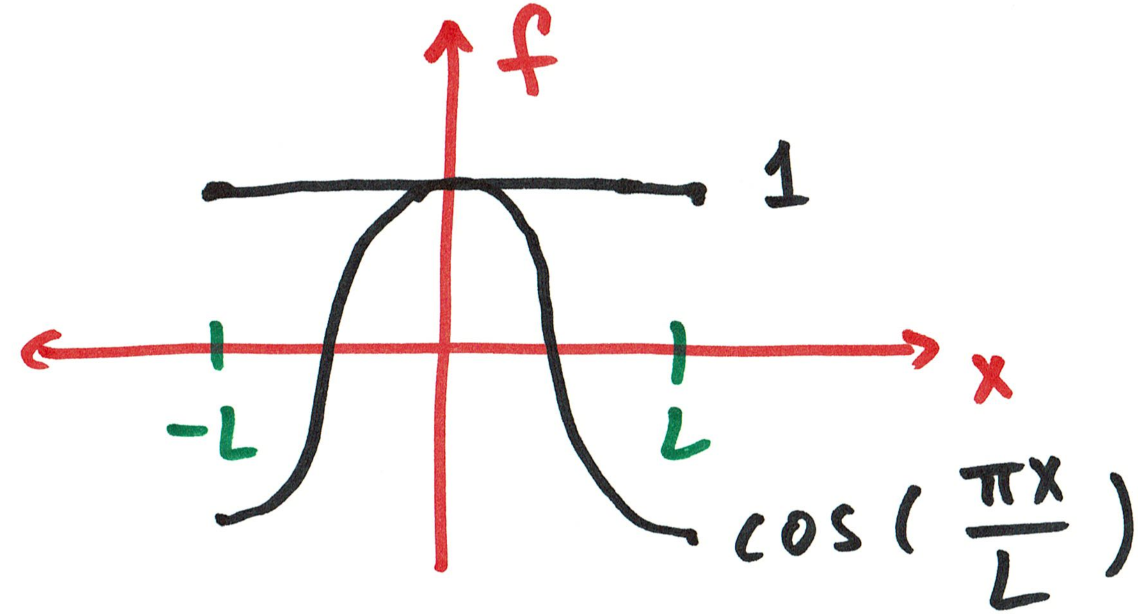
$$\|f\|^2 = \langle f, f \rangle = \int_{-L}^{L} f^2(x)\,dx = 2L$$

so $\|f\| = \sqrt{2L}$.

Two important subspaces

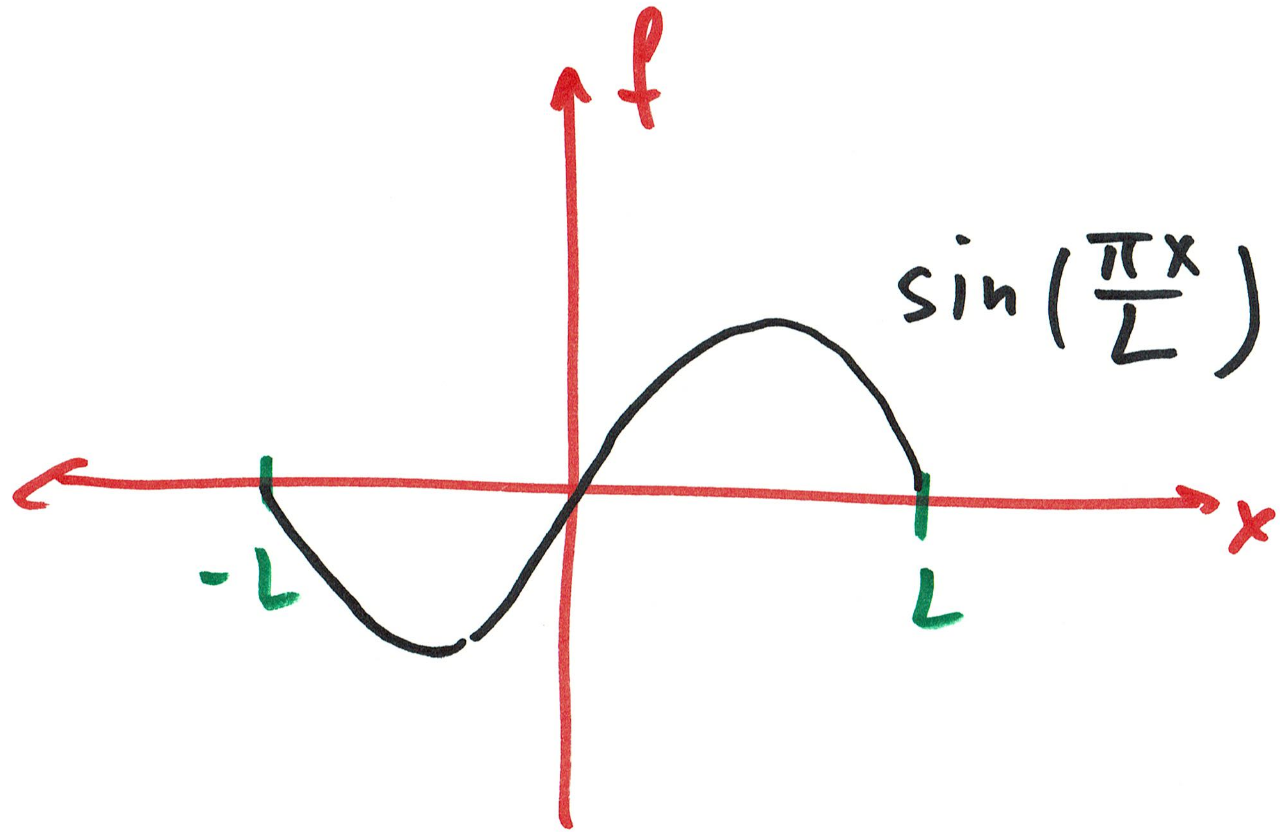
1) $V_L^{ev} = \{ f \in V_L, \ f(x) = f(-x) \}$

Ex $f(x) = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \ldots$



2) $V_L^{odd} = \{ f \in V_L, \ f(-x) = -f(x) \}$

<u>Ex</u> $f(x) = \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$



<u>Rmk</u> Can package all together

$$e^{i n\pi x/L} = \cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)$$

If you allow complex numbers then exponential gives fns in V_L.

__Lemma__ V_L^{ev}, V_L^{odd} are each other's orthog. complements

$$(V_L^{ev})^\perp = V_L^{odd}$$

$$(V_L^{odd})^\perp = V_L^{ev}$$

__Pf.__ $f \in V_L^{ev}$, $g \in V_L^{odd}$

$$\langle f, g \rangle = \int_{-L}^{L} \underbrace{f(x)g(x)}_{\text{odd!}}\,dx \Rightarrow = 0$$

So $V_L^{ev} \subset (V_L^{odd})^\perp$, $(V_L^{odd})^\perp \subset V_L^{ev}$

To see above <u>inclusions</u> are <u>equalities</u>

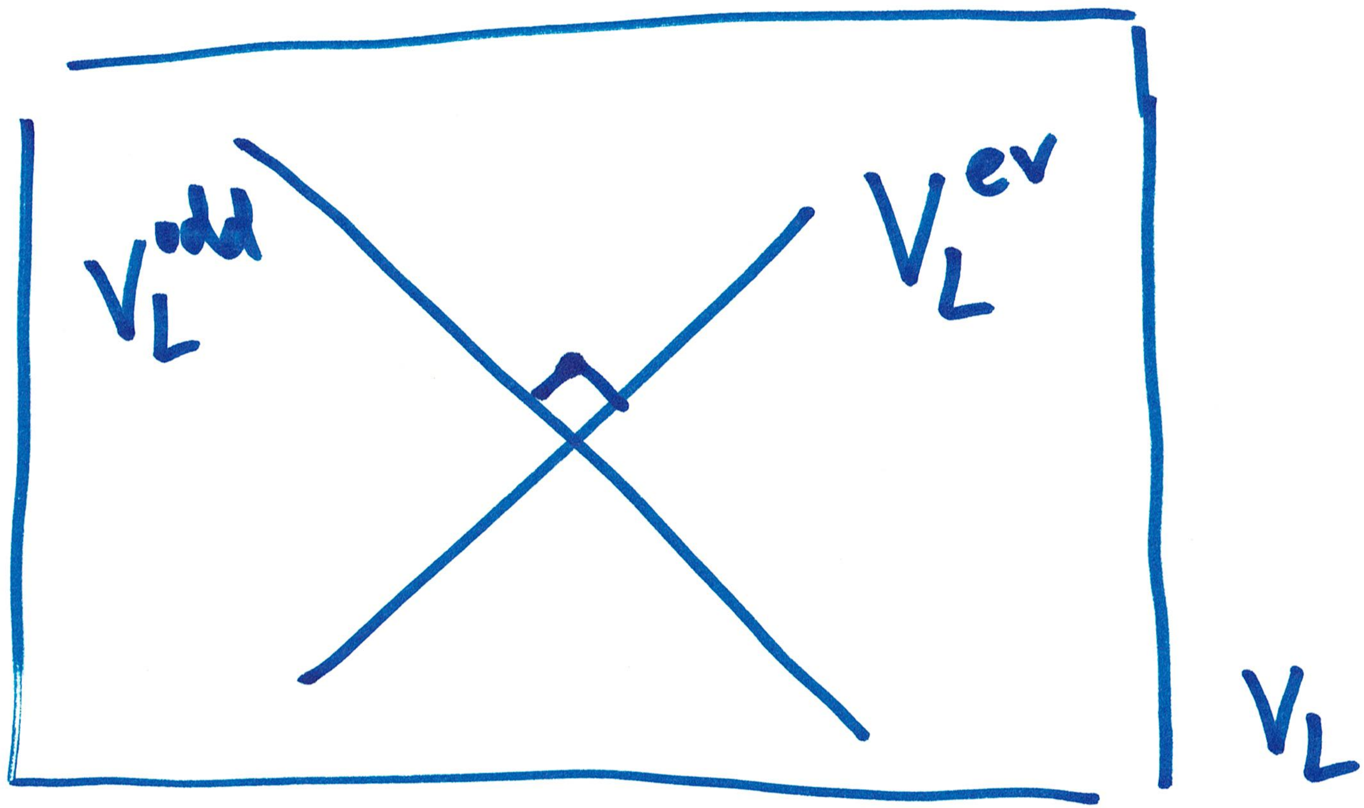
we can observe any fn $h \in V_L$

can be written uniquely as $h = h^{ev} + h^{odd}$

$$h^{ev}(x) = \frac{h(x) + h(-x)}{2}$$

$$h^{odd}(x) = \frac{h(x) - h(-x)}{2}$$

<u>Cartoon</u> of <u>Lemma</u>



Prop. $\cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, \ldots$

$\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$

form an **orthog set** so since they are nonzero, they are **lin. indep**!

__Pf.__ Need

1) $\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle = 0, \quad m \neq n$

2) $\left\langle \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = 0, \quad m \neq n$

3) $\left\langle \cos\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = 0, \quad \text{any } m, n$

3) is __easy__: inner prod of even and odd fns!

Let's check 2) together

$$\left\langle \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right)dx$$

trig id.

$$= \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

orthog ✓

$\left\| \sin\left(\frac{n\pi x}{L}\right) \right\| = \sqrt{L}$

3) is similar, check it!

Miraculous Observation $\cos\left(\frac{n\pi x}{L}\right)$, $n=0,1,2,\ldots$

$\sin\left(\frac{n\pi x}{L}\right)$, $n=1,2,3,\ldots$ "span" V_L

need: "∞-lin combs"

So they form "basis" in that they are lin indep & "span".

Let's write formulas for "**coords**" of vectors wrt this "basis".

__Def__ __Fourier__ __coeffs__ of $f \in V_L$ are

$$a_n = \frac{\langle f, \cos(\frac{n\pi x}{L}) \rangle}{\langle \cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \rangle} \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{\langle f, \sin(\frac{n\pi x}{L}) \rangle}{\langle \sin(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L}) \rangle} \qquad n = 1, 2, \ldots$$

Expanding out def...

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n=1,2,3,\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n=1,2,3,\ldots$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad \text{careful!}$$

Caution Textbook defines

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

This would result from taking first "basis" function to be

n=0 $\qquad \frac{1}{2}\cos\left(\frac{0\cdot\pi x}{L}\right) = \frac{1}{2}$

constant fns

instead of

n=0 $\qquad \cos\left(\frac{0\cdot\pi x}{L}\right) = 1$

__Def__ __Fourier series__ of $f \in V_L$

$$FS_f(x) = a_0 \cdot 1 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$+ \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

∞-"lin comb" of ~~orthog~~ __orthog__ __vectors__ ~~&~~

__Thm__ Any $f \in V_L$ with f' piece-wise continuous satisfies

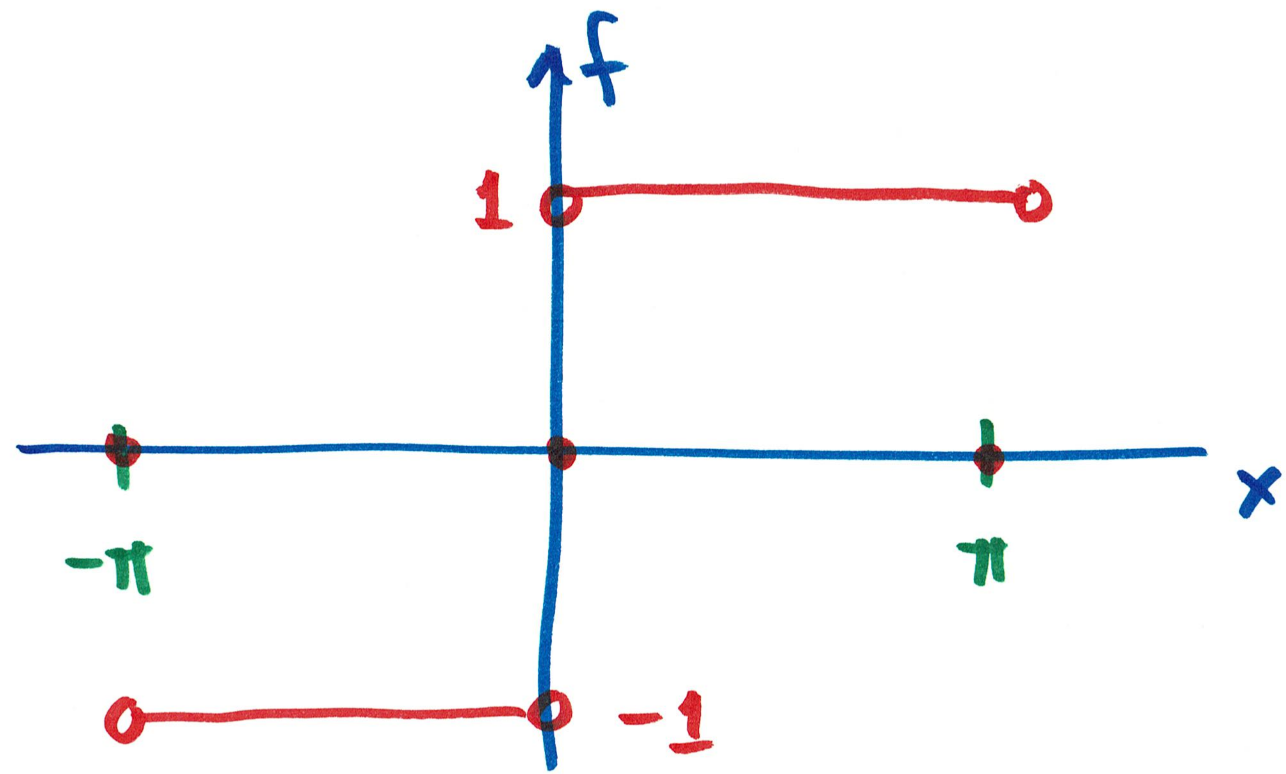
$$f(x) = FS_f(x) \quad \text{if } f \text{ cont. at } x$$

$$\left(\frac{f(x^+)+f(x^-)}{2} = FS_f(x) \quad \text{in general} \right)$$

Our orthog set "s<u>pan</u>s" !!!

<u>Exer</u> Set <u>$L=\pi$</u>. Calc FS_f for

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 0 & x = -\pi, 0, \pi \\ 1 & 0 < x < \pi \end{cases}$$



Note $f \in V_L^{odd}$ so $a_n = 0, \quad n = 0, 1, 2, \ldots$

$$FS_f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\right)\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\cdot\begin{cases} 0 & n \text{ even} \\ \frac{2}{n} & n \text{ odd} \end{cases}$$

Organize answer:

$$FS_f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots \right)$$

We're done!

What does Thm tell us?

$$f(x) = FS_f(x) \quad \text{where } f \text{ is cont: away from } k\pi,\ k \text{ integer}$$

Let's evaluate at $x = \frac{\pi}{2}$:

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots !$$