Lecture 24 Nonhomogenous lin systs of ODE

and a preview of coming attractions
if time permits

This week Happy Thanksgiving!

" For the Truth the Turkey is in Comparison a much more respectable Bird [than the Bald Eagle] ... though a little vain & silly, a Bird of Courage... "

Ben Franklin

__Recall__ Any nth order lin ODE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f$$

can be transformed into lin syst of 1st order ODE in __normal__ __form__

$$\underline{x}' = A \underline{x} + \underline{b}$$

matrix

vects of fns

Via substitution

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \begin{aligned} x_1 &= y \\ x_2 &= y^{(1)} \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

IVP: $y(0) = Y_0,\ y^{(1)}(0) = Y_1,\ \ldots,\ y^{(n-1)}(0) = Y_{n-1}$

transforms to requirement

$$\underline{x}(0) = \underline{Y} \qquad \underline{Y} = \begin{bmatrix} Y_0 \\ \vdots \\ Y_{n-1} \end{bmatrix}$$

We are experts at solving <u>homog</u> <u>version</u>

$$\underline{x}' = A\underline{x} \qquad (\underline{b} = \underline{0})$$

<u>Two strategies</u>

1) when A has basis of e-vectors $\underline{v}_1, \dots, \underline{v}_n$

with e-values $\lambda_1, \dots, \lambda_n$

then <u>basis</u> of <u>solns</u> $\underline{x}_1 = e^{\lambda_1 t}\underline{v}_1, \dots, \underline{x}_n = e^{\lambda_n t}\underline{v}_n$

Always fun, not always possible

2) form <u>matrix</u> expo<u>n</u>ential $X = e^{At}$

$$= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

<u>basis</u> of <u>solns</u> $\quad$ cols of $X = \begin{bmatrix} | & & | \\ \underline{x}_1 & \cdots & \underline{x}_n \\ | & & | \end{bmatrix}$

<u>fund soln matrix</u>

__Exer__ Find basis of solns for $\underline{x}' = A\underline{x}$

1) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

__Soln__ $\chi_A(t) = t^2 + 1$ __e-values__ : $i, -i$

__e-vectors__ $\begin{bmatrix} 1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}$

Basis of __solns__ $\underline{x}_1 = e^{it}\begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad \underline{x}_2 = e^{-it}\begin{bmatrix} 1 \\ i \end{bmatrix}$

Let's <u>also</u> try <u>2nd method</u>:

$$X = e^{At}, \quad \text{for } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} + \begin{bmatrix} -t^2/2 & 0 \\ 0 & -t^2/2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & t^3/6 \\ -t^3/6 & 0 \end{bmatrix} + \begin{bmatrix} t^4/24 & 0 \\ 0 & t^4/24 \end{bmatrix}$$

$$+ \cdots$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2} + \frac{t^4}{24} - \cdots & -t + \frac{t^3}{6} - \cdots \\ t - \frac{t^3}{6} + \cdots & 1 - \frac{t^2}{2} + \frac{t^4}{24} - \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

Alternative basis of solns $\quad \underline{y}_1 = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \quad \underline{y}_2 = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$

Exer Check $\text{Span}\{\underline{x}_1, \underline{x}_2\} = \text{Span}\{\underline{y}_1, \underline{y}_2\}$!

Hint: use Euler's formula!

2) $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

<u>Soln</u> <u>e-value</u>: 2 (mult = 3)

<u>Exer</u>: $\dim E_2 = \underline{1} < 3$

uh oh, <u>not</u> diagonalizable!

Try to calculate matrix exponential

$$X = e^{At}$$

Nice observation to reduce to finite calc possible since only single e-value

$$e^{(M+N)t} = e^{Mt} e^{Nt} \,!$$

check this!

<u>Apply</u> <u>with</u>: $M = 2I$, $N = A - 2I$

single e-value

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{Mt} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

e^{Nt} will have <u>finite</u> expansion since 0 is only e-value

We find

$$e^{Nt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 2t \\ 0 & 0 & -t \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} 0 & 0 & -t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots$$

all higher order terms $= 0$!

$$e^{Nt} = \begin{bmatrix} 1 & t & 2t - \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$

Now let's consider nonhomog case

$$\underline{x}' = A\,\underline{x} + \underline{b} \leftarrow \text{not } \underline{0}$$

Two approaches

1) undet coeffs (educated guesswork...)
2) var of params (integration ...)

Exer Find general soln to

$$\underline{x}' = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 3t \\ e^t \end{bmatrix}$$

Soln Homog version

$$\underline{x}' = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \underline{x}$$

Why are you happy?

$A = A^T$!

We find e-values $\lambda_1 = 2$, $\lambda_2 = -3$

e-vectors $\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

basis of solns $\underline{x}_1 = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\underline{x}_2 = e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Done with homog version!

First apply <u>superposition</u> <u>principle</u>

$$\underline{b} = \begin{bmatrix} -3t \\ e^t \end{bmatrix} = \underbrace{\begin{bmatrix} +3t \\ 0 \end{bmatrix}}_{\underline{f}} + \underbrace{\begin{bmatrix} 0 \\ e^t \end{bmatrix}}_{\underline{g}}$$

Solve $\underline{x}' = A\underline{x} + \underline{f}$, $\underline{x}' = A\underline{x} + \underline{g}$

<u>independently</u>

then add together solns.

<u>Focus</u> on $\underline{f} = \begin{bmatrix} 3t \\ 0 \end{bmatrix}$

<u>Method</u> of <u>undet coeffs</u>

try $\underline{x}_p = \underline{a}\,t + \underline{b}$

$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Solve for $\underline{a}$, $\underline{b}$...

Returning to eqn:

$$\underline{x}_p' = \underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

need //

$$A\underline{x}_p + \underline{f} = A(\underline{a}t + \underline{b}) + \underline{f}$$

$$\cancel{\forall t: A\underline{a}t + (A\underline{b} + \underline{f})}$$

<u>t terms</u> $\underline{0} = t \cdot A\underline{a} + \underline{f}$

lin syst
for $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $A\underline{a} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

since $f = \begin{bmatrix} 3t \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ 2 & -2 & 0 \end{array}\right]$ <u>Solns</u> $\underline{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
(unique)

<u>const terms</u> $\underline{x}_p' = \underline{a} = A\underline{b}$

lin syst
for $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\quad A\underline{b} = \underline{a}$

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & -2 & -1 \end{array}\right]$$

solu $\underline{b} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{6} \end{bmatrix}$
(unique)

We're done!

$$\underline{x}_p = \underline{a}t + \underline{b} = \begin{bmatrix} -t - 2/3 \\ -t - 1/6 \end{bmatrix}$$

Gen'l soln: $\underline{x} = \underline{x}_p + c_1 \underline{x}_1 + c_2 \underline{x}_2$

to $\underline{x}' = A\underline{x} + \underline{f}$

Exer Similarly solve

$$\underline{x}' = A\underline{x} + \underline{g}, \quad \underline{g} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

Try $x_p = e^t \underline{a}$, $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

solve for $\underline{a}$ ....

What form does var of params take?

Try $\underline{x}_p = X \, \underline{v}$

find matrix of solns for homog eqn

vect of fns.

(Should remind you of

$x_p = v_1 x_1 + v_2 x_2 \ldots$ for 2nd order ODE)

Similar derivation yields

$$\underline{v} = \int X^{-1} \underline{b} \, dt$$

so

$$\underline{x}_p = \underbrace{\underset{\text{matrix}}{X} \underbrace{\int X^{-1} \underline{b} \, dt}_{\text{vector}}}_{\text{vector}}$$

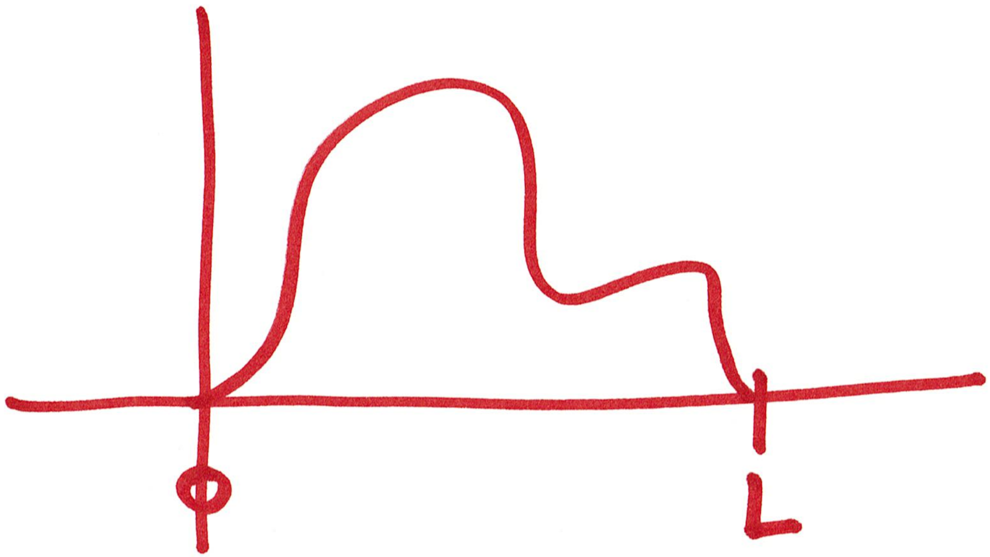
<u>In</u> <u>prior exercise</u> $X = \begin{bmatrix} 2e^{2t} & -e^{-3t} \\ e^{2t} & 2e^{-3t} \end{bmatrix}$

$$\underline{b} = \begin{bmatrix} 3t \\ e^{t} \end{bmatrix}$$

so $\underline{x}_p = X \int X^{-1} \underline{b} \, dt$

for above quantities.

Preview of coming attractions



$$V = \{ f : [0,L] \to \mathbb{R} \mid f(0) = f(L) = 0 \}$$

"Basis" of fns

$$\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

"Basis" is in quotes since we will use <u>∞-sums</u>

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$