<u>Lecture 22</u> Undetermined Coeffs vs Variation of Parameters

Let's get ready to rumble!

<u>Wed</u> Office Hours are back! 12-2pm, 891 Evans

<u>Fri</u> Quiz through § 9.1 NSS

"Service to others is the rent you pay for your room here on earth."
Muhammad Ali

Round 1 Find a soln to

$$y'' - y' - 2y = e^{2t}$$

(Focus for today: finding a soln to nonhomog 2nd order lin ODE)

Homog eqn $\quad y'' - y' - 2y = 0$

aux eqn $\quad r^2 - r - 2 = 0$

factors $\quad (r-2)(r+1) = 0$

roots $\quad r_1 = 2,\ r_2 = -1$

Basis of solns to homog eqn $y_1 = e^{r_1 t} = e^{2t}, \quad y_2 = e^{r_2 t} = e^{-t}$

<u>Back to nonhomog</u> eqn

<u>Undet coeffs</u> $f = e^{2t}$ ← is root of aux eqn

so try $y_p = a t e^{2t}$

<u>To find</u> a :

$$y_p' = a e^{2t} + 2 a t e^{2t}$$

$$y_p'' = 2 a e^{2t} + 2 a e^{2t} + 4 a t e^{2t}$$

$$= 4 a e^{2t} + 4 a t e^{2t}$$

Subst back into eqn $y_p'' - y_p' - 2y_p = 3ae^{2t}$

Since $f = e^{2t}$, conclude take $a = \frac{1}{3}$

So $y_p = \frac{1}{3} t e^{2t}$

Now try <u>Variation</u> of <u>Parameters</u>

Try $y_p = v_1 y_1 + v_2 y_2 = v_1 e^{2t} + v_2 e^{-t}$

functions of t!

<u>To find</u> v_1, v_2

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

$$= v_1 y_1' + v_2 y_2' + \boxed{v_1' y_1 + v_2' y_2}$$

Let's assume $\boxed{v_1' y_1 + v_2' y_2 = 0 \quad (*)}$

to avoid later appearance of 2nd derivs of v_1, v_2

Given $(*)$, we have $y_p' = v_1 y_1' + v_2 y_2'$

so $y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$

Subst back into eqn ... $y_p'' + b y_p' + c\, y_p = f$

implies $\boxed{v_1' y_1' + v_2' y_2' = f \quad (**)}$

Now use (*) + (**) to solve for v_1, v_2

Key pt only 1st derivs appear!

Observe (*) + (**) are "lin syst" for v_1', v_2'

$$\begin{array}{c} (*) \\ \\ (**) \end{array} \left[\begin{array}{cc|c} y_1 & y_2 & 0 \\ & & \\ y_1' & y_2' & f \end{array}\right]$$

Wow!

"lin syst" = lin syst for each t

__Note__ Coeff matrix $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$

is invertible for all t

since $W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ Wronskian

is __never__ 0 for any t since

y_1, y_2 basis
of solnc to
homog eqn

So to solve "lin syst": invert coeff matrix and mult it against avg. vect.

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$= \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix}$$

$W = y_1 y_2' - y_2 y_1'$

__Conclude__

$$v_1' = \frac{-y_2 f}{y_1 y_2' - y_2 y_1'}$$

$$v_2' = \frac{y_1 f}{y_1 y_2' - y_2 y_1'}$$

__Good__ news : find v_1, v_2 by integration!

Bad __news__: integration... 

<u>Back to exer</u> $y_1 = e^{2t}, \; y_2 = e^{-t}, \; f = e^{2t}$

$$W = e^{2t}(-e^{-t}) - e^{-t}(2e^{2t})$$

$$= -3e^{t}$$

<u>So find:</u> $v_1 = \int \frac{-e^{t}}{-3e^{t}}\,dt = \int \frac{1}{3}\,dt$

$$= \frac{t}{3} + c_1$$

$$v_2 = \int \frac{e^{4t}}{-3e^{t}}\,dt = \int -\frac{1}{3}e^{3t}\,dt = -\frac{1}{9}e^{3t} + c_2$$

Grand conclusion

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(\frac{1}{3}t + c_1\right)e^{2t} + \left(-\frac{1}{9}e^{3t} + c_2\right)e^{-t}$$

$$= \underbrace{\frac{1}{3}te^{2t}}_{\text{prev. found soln to nonhomog eqn!}} + \underbrace{\left(c_1 - \frac{1}{9}\right)e^{2t} + c_2 e^{-t}}_{\text{soln to homog eqn}}$$

Take $c_1 = c_2 = 0$ to find single soln.

<u>Round 1 winner</u> Undet coeffs
by knockout!

<u>On to Round 2</u> Find a soln to

$$y'' + y = \operatorname{cosec}(t)$$

<u>Undet</u> coeffs: ¯_(")_/¯

Var <u>of params</u> homog eqn $y'' + y = 0$

aux eqn $r^2 + 1 = 0$ roots $r_1 = i$, $r_2 = -i$

basis of solns for homog eqn $y_1 = e^{it}$, $y_2 = e^{-it}$

Since $f = \mathrm{cosec}(t) = \dfrac{1}{\sin(t)}$, convenient to use <u>real basis</u> $u_1 = \mathrm{Re}(y_1) = \cos(t)$

$u_2 = \mathrm{Im}(y_1) = \sin(t)$

Same route as before gives

$$v_1' = \frac{-u_2 f}{u_1 u_2' - u_2 u_1'} \qquad v_2' = \frac{u_1 f}{u_1 u_2' - u_2 u_1'}$$

so $\quad v_1 = \int \quad dt \qquad v_2 = \int \quad dt$

Back to specific fns: $u_1 = \cos,\ u_2 = \sin$

$f = \frac{1}{\sin},\ W = 1$

$$v_1 = \int -\frac{\sin(t)}{\sin(t)}\,dt = -t + c_1$$

$$v_2 = \int \frac{\cos(t)}{\sin(t)}\,dt = \ln|\sin(t)| + c_2$$

$\cot(t)$

Conclusion $y_p = v_1 u_1 + v_2 u_2$

$$= (-t + c_1)\cos(t) + (\ln|\sin(t)| + c_2)\sin(t)$$

Set $c_1 = c_2 = 0$ to find a single soln

$$y_p = -t\cos(t) + \ln|\sin(t)|\sin(t)$$

Round 2 to Var of Params!

Wait a second! Have you taught us anything new?!?...

Claim All nth order lin ODE can be written as a system of 1st order lin ODE!

Ex $\quad y'' + by' + cy = f$

Set: $\quad x_1 = y$

$x_2 = y'$ ← think of as an independent fn!

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ vector of fns.

Then

$$\underline{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} =$$

$$= \begin{bmatrix} y' \\ -by' - cy + f \end{bmatrix}$$

$$\underline{x}' = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \underline{x} + \underline{b} \qquad \underline{b} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$