Lecture 19 Singular Value Decomposition!

Fri Quiz through §7.2

Raindrops on roses
And whiskers on kittens
Singular value decompositions
Brown paper packages tied up with strings
These are a few of my favorite things

Rodgers & Hammerstein

Next week Diff Eq!

Recall main goal of e-vectors, e-values:

given $n \times n$ matrix A

find basis $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ of e-vectors

if possible

form $P = P_\beta = P_{\varepsilon \leftarrow \beta} = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$

diagonalization $D = P_\beta^{-1} A P_\beta$

$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ e-values $\lambda_1, \dots, \lambda_n$

Possible if:

1) "enough e-vectors" $\dim E_\lambda$ = mult of λ in $\chi_A(t)$

or 2) $\underline{A}$ symmetric: $A = A^T$

in fact can choose orthon. basis

$$\beta = \{\underline{v}_1, \ldots, \underline{v}_n\}$$

so that $P = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$ orthog matrix

"rotations, reflections, ..."

New question

given $m \times n$ matrix A

find bases $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ of $\mathbb{R}^n$

$\gamma = \{\underline{w}_1, \dots, \underline{w}_m\}$ of $\mathbb{R}^m$

so that

$$\Sigma = Q_\gamma^{-1} A P_\beta$$

is as "nice" as possible

__Bonus__ what if bases are orthon. so that P_β, Q_γ are orthog.?

__Rmk__ Question is equiv. to asking what form can we achieve with row ops (Q_γ^{-1}) __and__ col ops (P_β)?

<u>Soln</u> Set $r = \mathrm{rk}(A)$.

We will achieve

$$\Sigma = \begin{array}{cc} \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} \}\, r \\ \}\, m-r \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{r} & \underbrace{\qquad}_{n-r} \end{array} & \end{array}$$

"Up to row and col ops,
all $\mathrm{rk} = r$ lin transfs
are the same!"

Pf Choose basis $\underline{x}_1, \ldots, \underline{x}_k$ of $\mathrm{Null}(A) \subset \mathbb{R}^n$

Complete to basis $\beta = \{\underline{v}_1, \ldots, \underline{v}_r, \underline{x}_1, \ldots, \underline{x}_k\}$ of $\mathbb{R}^n$

$k + r = n$

Exer Show $\underline{w}_1 = A\underline{v}_1, \ldots, \underline{w}_r = A\underline{v}_r$ are <u>lin</u> <u>indep</u> in $\mathbb{R}^m$

Complete to basis $\gamma = \{\underline{w}_1, \ldots, \underline{w}_r, \underline{y}_1, \ldots, \underline{y}_\ell\}$ of $\mathbb{R}^m$

$\ell + r = m$

Check matrix of A w.r.t. bases β, γ has form

$$\Sigma = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

Why?

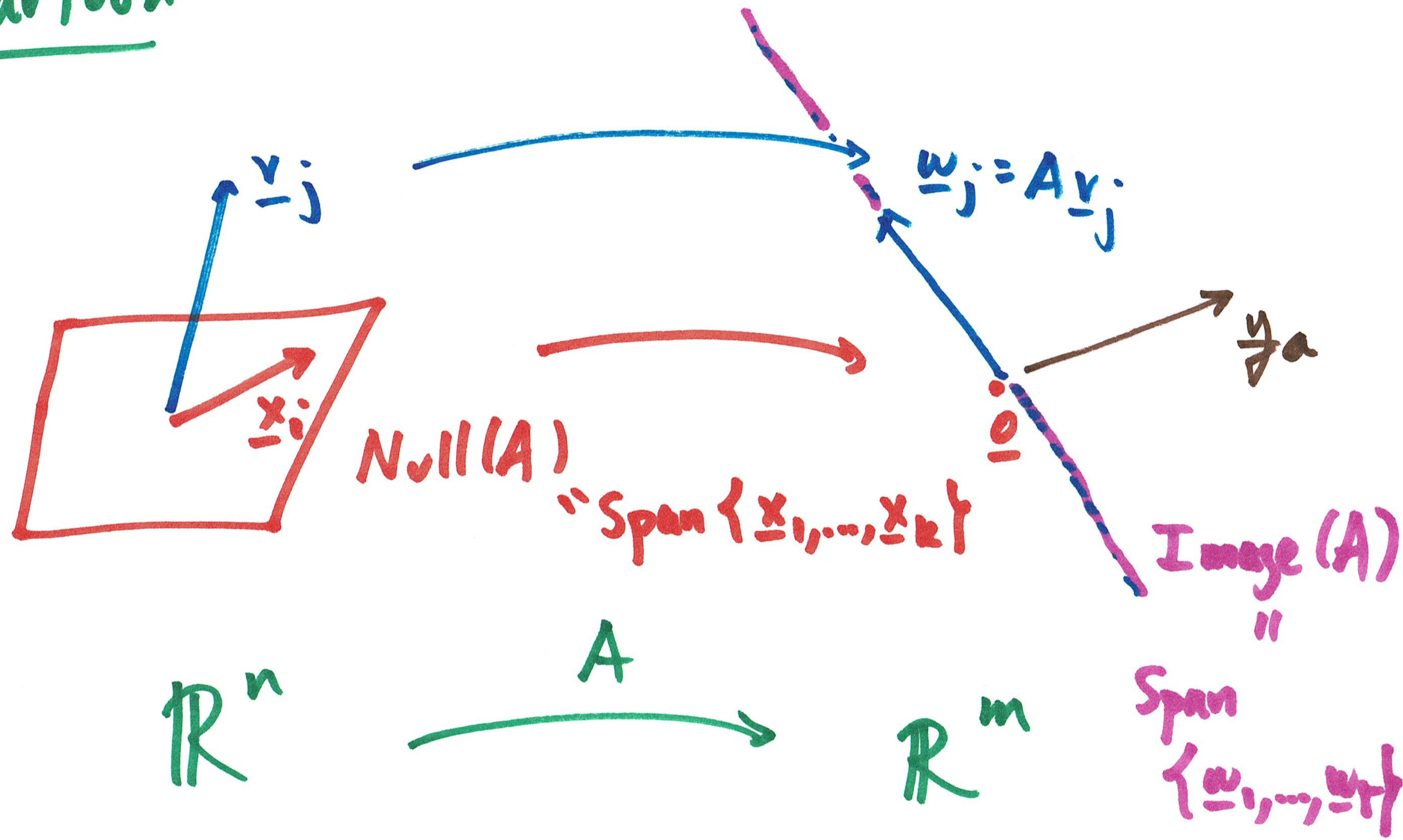
$$\Sigma = \begin{bmatrix} | & & | & | & & | \\ [A\underline{v}_1]_\gamma & \cdots & [A\underline{v}_r]_\gamma & [A\underline{x}_1]_\gamma & \cdots & [A\underline{x}_k]_\gamma \\ | & & | & | & & | \end{bmatrix}$$

$$
= \left[\begin{array}{cccc|c}
1 & 0 & & & \\
0 & 1 & \ddots & & 0 \\
\vdots & 0 & & & \\
\vdots & \vdots & & 1 & \\
\hline
& \vdots & & & \\
0 & 0 & \cdots & 0 & 0
\end{array}\right]
$$

$\underline{w}_i = A\underline{v}_i$ $\qquad$ $A\underline{x}_j = \underline{0}$

Done!

Cartoon


$\underline{v}_j$
$\underline{w}_j = A\underline{v}_j$
$\underline{y}_a$
$\underline{x}_i$
Null(A)
= Span$\{\underline{x}_1, \dots, \underline{x}_k\}$
$\underline{0}$
Image(A)
=
Span
$\{\underline{w}_1, \dots, \underline{w}_r\}$
A
$\mathbb{R}^n$
$\mathbb{R}^m$

Ex Find β, γ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \end{bmatrix}$

$$\rightsquigarrow A_{ref} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -7 \end{bmatrix} \quad rk = 2$$

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \ldots \quad \begin{array}{l} \beta = ? \\ \gamma = ? \end{array}$$

Basis for Null(A): $\underline{x}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{7}{4} \\ 1 \end{bmatrix}$

Complete to basis of $\mathbb{R}^3$

$$\beta = \left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{x}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{7}{4} \\ 1 \end{bmatrix} \right\}$$

Apply A to find basis for Image(A):

$$\underline{w}_1 = A\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \underline{w}_2 = A\underline{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Complete to basis of $\mathbb{R}^2$ already a basis!

$$\gamma = \left\{ \underline{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \underline{w}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$$

<u>Bonus</u> What if we insist β, γ are <u>orthon</u> bases so that P_β, Q_γ are <u>orthog</u>?

<u>S.V.D. Thm</u> A $m \times n$ matrix

There are orthon. bases $\beta = \{\underline{v}_1, \ldots, \underline{v}_n\}$
$\gamma = \{\underline{w}_1, \ldots, \underline{w}_m\}$

so that

$$\Sigma = Q_\gamma^{-1} A P_\beta$$

unique

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

$$= \left[\begin{array}{c|c} \begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

Bases β, γ are not nec unique

but $\sigma_1, \ldots, \sigma_r$ are <u>unique</u>

<u>Def</u> $\sigma_1, \ldots, \sigma_r$ are (non-zero) <u>sing</u> <u>values</u> of A

How to find sing values?

Consider $A^T A$

$(n \times m)\ (m \times n)$

$(n \times n)$ square!

Symmetric! $(A^T A)^T = A^T (A^T)^T$

$= A^T A$

Spectral Thm: find orthon. basis $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ of e-vectors of $A^T A$

this will be our domain basis for SVD!

__Claim__ E-values $\lambda_1, \dots, \lambda_n$ of A^TA are all ≥ 0 !

__Why?__

$$\underbrace{\langle A\underline{v}_i, A\underline{v}_i \rangle}_{\geq 0} = \underline{v}_i^T \underbrace{A^T A}\, \underline{v}_i$$

$$= \lambda_i \underline{v}_i^T \underline{v}_i$$

$$= \lambda_i \underbrace{\langle \underline{v}_i, \underline{v}_i \rangle}_{> 0}$$

Conclude $\lambda_i \geq 0$!

Reorder e-vectors, e-valve so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

Set : $\sigma_1 = \sqrt{\lambda_1} \geq \cdots \geq \sigma_r = \sqrt{\lambda_r} > 0$

pos sqr-root of (non-zero) e-values of $A^T A$

<u>Now</u> <u>pf</u> <u>of SVD thm</u>

Take $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ orthon. basis of $\mathbb{R}^n$

(coming from orthog. diag. of $A^T A$)

Take $\underline{w}_1 = \dfrac{A\underline{v}_1}{\sigma_1}, \dots, \underline{w}_r = \dfrac{A\underline{v}_r}{\sigma_r}$

<u>Claim</u> $\underline{w}_1, \dots, \underline{w}_r$ is orthon. basis of Image(A)

<u>Why</u>? $\langle \underline{w}_i, \underline{w}_j \rangle = \left\langle \frac{A\underline{v}_i}{\sigma_i}, \frac{A\underline{v}_j}{\sigma_j} \right\rangle$

$$= \frac{1}{\sigma_i \sigma_j} \underline{v}_i^T \underbrace{A^T A} \underline{v}_j$$

$$= \frac{\lambda_j}{\sigma_i \sigma_j} \underline{v}_i^T \underline{v}_j = \frac{\lambda_j}{\sigma_i \sigma_j} \langle \underline{v}_i, \underline{v}_j \rangle$$

$$= \begin{cases} \dfrac{\lambda_i}{\sigma_i^2} = 1 & i = j \\ 0 & i \neq j \end{cases}$$

since $\langle \underline{v}_i, \underline{v}_j \rangle = 0$

<u>Check</u> $\underline{w}_1, \ldots, \underline{w}_r$ span Image(A).

Complete to orthon basis

$$\gamma = \{ \underbrace{\underline{w}_1, \ldots, \underline{w}_r}_{\substack{\text{orthon. basis} \\ \text{for Image}(A)}}, \underbrace{\underline{w}_{r+1}, \ldots, \underline{w}_m}_{\substack{\text{orthon basis} \\ \text{for Image}(A)^\perp}} \}$$

__Check__ $\Sigma = Q_\gamma^{-1} A P_\beta$

$$= \left[\begin{array}{c|c} \begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] \; !$$

$\underline{w}_i = A\underline{v}_i$

A applied to Null(A)

Tada!

__Ex__ Find SVD of $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

__Soln__ Find sing values = $\sqrt{\ }$ of pos e-values of A^TA

$$A^TA = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$\ldots\ \lambda = 6\ ,\ 0$ $\rightsquigarrow$ sing values

mult $= 1\ ,\ 2$

$\sigma_1 = \sqrt{6}$

Orthon.

Basis of e-vectors in $\mathbb{R}^3$

$$\beta = \left\{ \underline{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \underline{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}$$

Need <u>orthon</u>

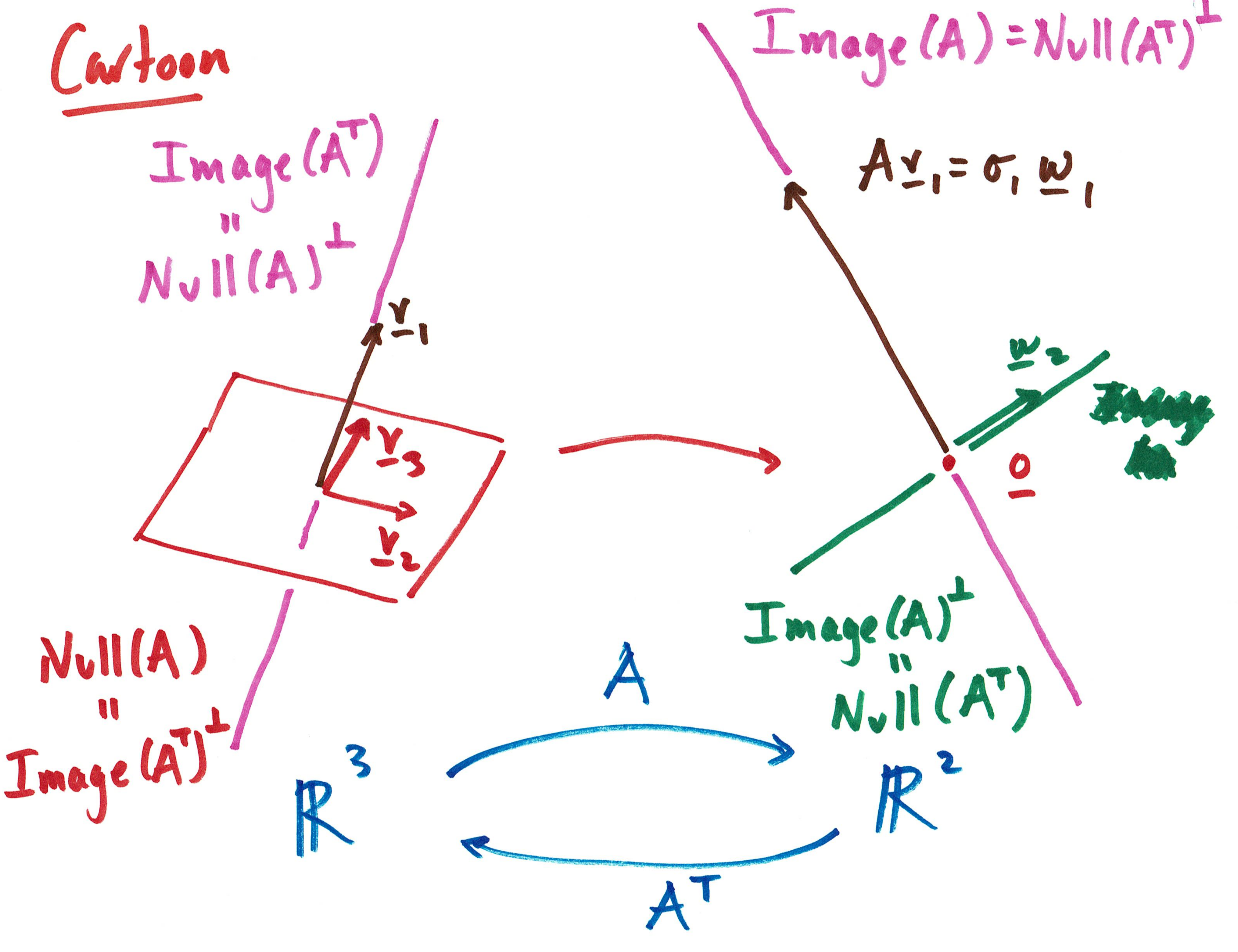
<u>Basic of $\mathbb{R}^2$</u>

$$\underline{w}_1 = \frac{A\underline{v}_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

complete to orthon basis ...

$$\underline{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

Cartoon



$\text{Image}(A) = \text{Null}(A^T)^\perp$
$A\underline{v}_1 = \sigma_1 \underline{w}_1$
$\text{Image}(A^T) = \text{Null}(A)^\perp$
$\underline{v}_1$
$\underline{v}_3$
$\underline{v}_2$
$\underline{w}_2$
$\underline{0}$
$\text{Null}(A) = \text{Image}(A^T)^\perp$
$\text{Image}(A)^\perp = \text{Null}(A^T)$
$\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^2$
A^T