Lecture 18

Beyond $\mathbb{R}^n$ and the dot product!

Today Midterm review, 2 - 3:30pm, 740 Evans

Fri Quiz through §7.1

Tues **10/31** Midterm 2 through §6.5

"To infinity and beyond!" Buzz Lightyear

__Recall__ __Spectral__ Thm A $n \times n$ matrix (with real entries)

TFAE

1) A __sym__ : $A = A^T$
2) There is an __orthog__ __basis__ of e-vectors of A
3) A is __diagonalizable__ $D = P^{-1}AP$ with P __orthogonal__

__Caution__ $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ has e-values $\lambda = 1, -2$ but their ~~eg~~-spaces E_1, E_2 are __not__ orthog

Remains to prove (1) $\Rightarrow$ (2)

By induction on n (= size of matrix)

Base case $n = 1$ $\quad A = [a]$

Take $\underline{v}_1 = [1]$, $\beta = \{\underline{v}_1\}$ orthog basis of e-vectors

Inductive step Assume true for $(n-1) \times (n-1)$ matrices

Consider some $n \times n$ sym matrix:

$$A = A^T$$

First goal: find e-vector!

Fund Thm of Alg $\Rightarrow$ $\chi_A(t)$ can be factored if we accept complex roots

Let $\lambda = a+ib$ be a root.

Claim λ is real $\lambda = a$

Why? Suppose not. Set $\lambda_+ = \lambda = a+ib$, $\lambda_- = \bar{\lambda} = a-ib$

Suppose $\underline{v}_+$ is an e-vector for λ_+

Then $\underline{v}_- = \overline{\underline{v}_+}$ is one for $\lambda_- = \bar{\lambda}_+$

Calc. $\lambda_+ (\underline{v}_+ \cdot \underline{v}_-) = (\lambda_+ \underline{v}_+) \cdot \underline{v}_-$

$$= (A\underline{v}_+) \cdot \underline{v}_- = (A\underline{v}_+)^T \underline{v}_-$$

$$= \underline{v}_+^T A^T \underline{v}_- \overset{!}{=} \underline{v}_+^T A \underline{v}_-$$

$$= \underline{v}_+ \cdot (A\underline{v}_-) = \underline{v}_+ \cdot (\lambda_- \underline{v}_-)$$

$$= \lambda_- (\underline{v}_+ \cdot \underline{v}_-)$$

Exer: Show $\underline{w} \cdot \overline{\underline{w}} > 0$ if $\underline{w} \neq \underline{0}$.

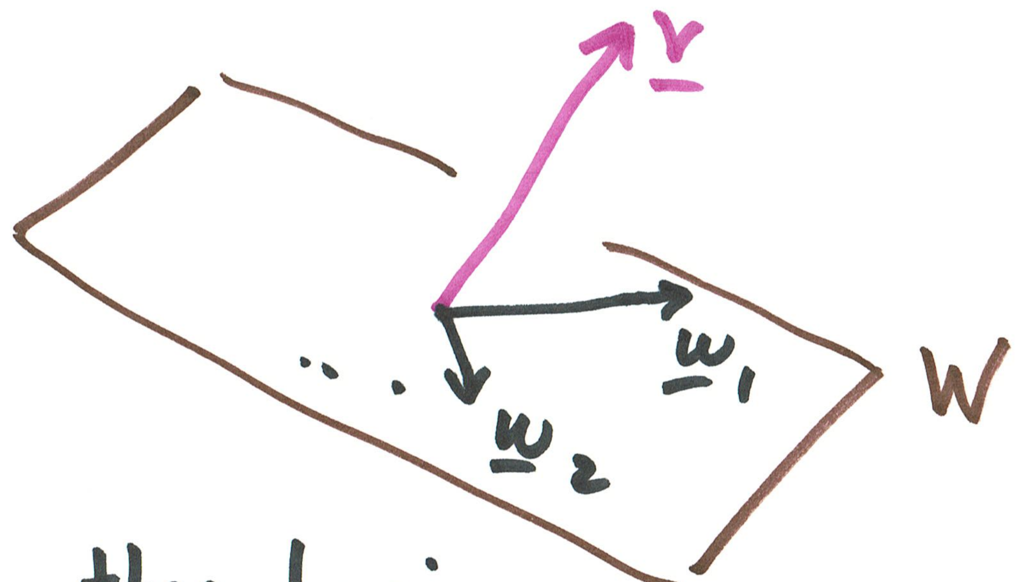
Conclude: $\lambda = \lambda_+ = \lambda_- = \bar{\lambda}$! so λ is real!

Exer λ real $\Rightarrow$ there is real e-vector $\underline{v}$

Why? $\mathrm{Null}(A-\lambda I) \neq \{\underline{0}\}$

Take some $\underline{v} \neq \underline{0}$ in null space. ∎

Now consider $W = \mathrm{Span}\{\underline{v}\}^{\perp}$



Choose orthog basis $\underline{w}_1, \ldots, \underline{w}_{n-1}$ for W

Consider basis $\beta = \{ \underline{w}_1, \ldots, \underline{w}_{n-1}, \underline{v} \}$ of $\mathbb{R}^n$

<u>Exer</u> $[A]_\beta = P_\beta^{-1} A P_\beta$

matrix of A w.r.t. basis β

is of form

$$\left[\begin{array}{c|c} \underset{(n-1)\times(n-1)}{A'} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & \lambda \end{array}\right]$$

with $(A')^T = A'$!

By <u>induction</u>, we can choose orthog basis $\underline{u}_1, \dots, \underline{u}_{n-1} \in \mathbb{R}^{n-1}$ of e-vectors for A'

<u>Define</u> $\underline{v}_1, \dots, \underline{v}_{n-1} \in \mathbb{R}^n$ by adding a single 0 component at end

$$\underline{v}_i = \begin{bmatrix} | \\ \underline{u}_i \\ | \\ 0 \end{bmatrix}$$

<u>Check</u> $\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{v}$ is orthog basis of e-vectors! ∎

Now let's go beyond $\mathbb{R}^n$ and dot product

Def An inner product space is a vect sp. V with an inner product, i.e. a function

$$\underset{\substack{\cap \\ V}}{\underline{u}, \underline{v}} \rightsquigarrow \underset{\substack{\cap \\ \mathbb{R}}}{\langle \underline{u}, \underline{v} \rangle}$$

satisfying :

1) $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$ (symmetric)

2) $\langle \underline{u}_1 + \underline{u}_2, \underline{v} \rangle = \langle \underline{u}_1, \underline{v} \rangle + \langle \underline{u}_2, \underline{v} \rangle$

(dist. over add.)

3) $\langle c\underline{u}, \underline{v} \rangle = c \langle \underline{u}, \underline{v} \rangle$ (scales)

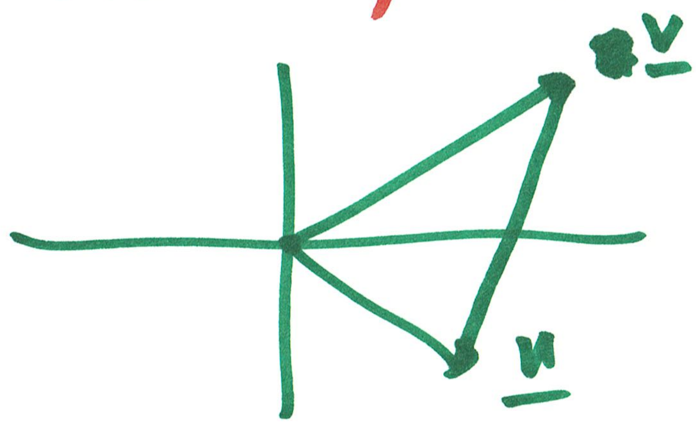
for real c

4) $\langle \underline{u}, \underline{u} \rangle \geq 0$ and

$= 0 \iff \underline{u} = \underline{0}$

Meta-principle: All notions for $\mathbb{R}^n$ with dot product work in any inner prod space.

For example: lengths, distances, angles, orthogonality, Pythag. Thm, law of cosines, Δ-inequality



$$\|\underline{u}-\underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|$$

Gram-Schmidt, orthog projs,

Cauchy-Schwarz inequality

$$|\langle \underline{u}, \underline{v} \rangle| \le \|\underline{u}\| \, \|\underline{v}\|$$

follows from identity

$$\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \, \|\underline{v}\| \cos\theta$$

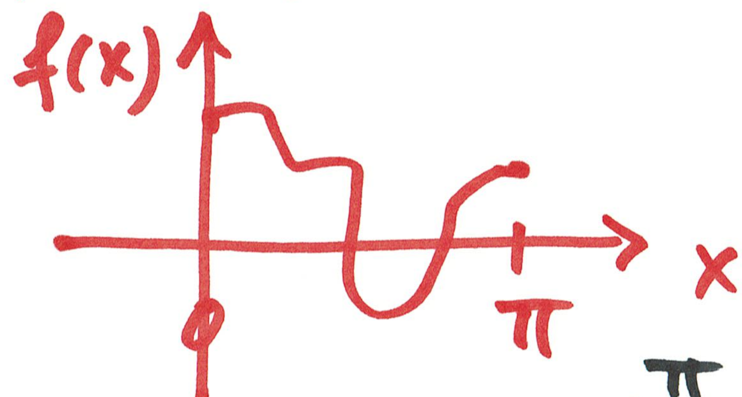
angle between $\underline{u}, \underline{v}$

...

Examples

1) (Beyond $\mathbb{R}^n$ ...)

<u>vect sp</u>: $V = \{ \text{cont. fns } f : [0,\pi] \to \mathbb{R} \}$



<u>inner</u> <u>prod</u>: $\langle f, g \rangle = \int_0^{\pi} f(x) g(x)\, dx$

Check axioms are satisfied!

Exer Calc. $\|\cos(x)\| = \sqrt{\langle \cos(x), \cos(x) \rangle}$

$$= \sqrt{\int_0^{\pi} \cos^2(x)\, dx}$$

$$= \sqrt{\int_0^{\pi} \left(\frac{1}{2} + \frac{\cos(2x)}{2} \right) dx}$$

$$= \sqrt{\frac{x}{2} + \frac{\sin(2x)}{4} \Big|_0^{\pi}}$$

$$= \sqrt{\frac{\pi}{2}}$$

2) (Beyond dot product ...)

$V = \mathbb{R}^n$ from here on

How to construct other inner prods?

A sym $n \times n$ matrix

Define $\langle \underline{u}, \underline{v} \rangle_A = \underline{u}^T A \underline{v}$

$$= \underline{u} \cdot (A\underline{v})$$

$$= (A\underline{u}) \cdot \underline{v}$$

__Note__ $A = \underline{I}_n \implies \langle \underline{u}, \underline{v} \rangle_A = \underline{u} \cdot \underline{v}$

__But__ ... what if $A = 0$???

Then $\langle \underline{u}, \underline{v} \rangle_A = 0$ always.

still satisfies axioms 1), 2), 3)

but __not__ 4)

__In general__ ~~may~~ $\langle \underline{u}, \underline{v} \rangle_A$ will satisfy axioms 1), 2), 3) but only __sometimes__ 4)

Further remark: If A arbitrary, not nec. sym., then axioms 2), 3) still hold but not 1).

Terminology **quadratic form** is a fn $Q : \mathbb{R}^n \to \mathbb{R}$ of the form $Q(\underline{v}) = \langle \underline{v}, \underline{v} \rangle_A$ for sym matrix A

"length-squared for some possible inner prod"

More terminology: a quad form Q is

1) pos-definite if $Q(\underline{v}) > 0$ for $\underline{v} \neq \underline{0}$

[Note: $\langle \underline{u}, \underline{v} \rangle_A$ is inner prod $\iff Q = \langle \underline{v}, \underline{v} \rangle_A$ is pos-def]

2) neg-definite $\cdots\cdots$ $Q(\underline{v}) < 0$

3) indefinite if sometimes $Q(\underline{v}) > 0$, sometimes $Q(\underline{v}) < 0$

For what A sym, is $\langle \underline{u}, \underline{v} \rangle_A$ an <u>inner prod</u>, i.e. $Q(\underline{v}) = \langle \underline{v}, \underline{v} \rangle_A$ <u>pos-def</u>?

Apply Spec<u>tra</u>l <u>Theorem</u>!

Find orthog P so that

$$D = P^{-1} A P \quad \text{equiv} \quad A = P D P^{-1}$$

diagonal

What can D look like?

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$\langle \underline{u}, \underline{v} \rangle_A$ is inner prod, i.e. $Q(\underline{v}) = \langle \underline{v}, \underline{v} \rangle_A$ is pos-def

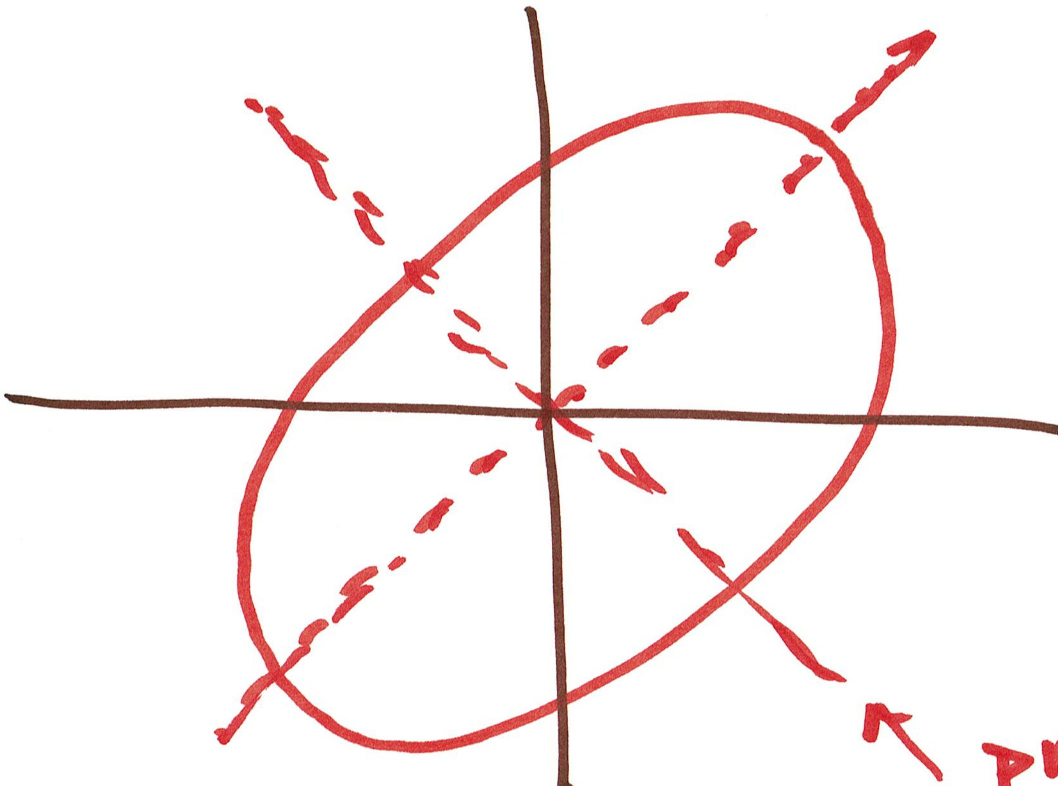
$$\Updownarrow$$

all $\lambda_i > 0$ for all $i = 1, \dots, n$

Ex $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

unit circle

is an ellipse



principle axes are eg-lines!