<u>Lecture 16</u> Orthogonality!

<u>Fri</u> Quiz through §6.3

<u>Next</u> <u>week</u> Extra Office Hours
Thurs, 2:00-3:30 pm, 740 Evans

<u>Midterm 2</u> Tues, Oct 31,
during lecture,
through §6.5

Boo!

Recall To do geom. in $\mathbb{R}^n$

(lengths, angles, ...)

we used dot product

$$\underline{u} \cdot \underline{v} = \underline{u}_1 \underline{v}_1 + \dots + \underline{u}_n \underline{v}_n$$

Question What $A : \mathbb{R}^n \to \mathbb{R}^n$ lin transf preserves these quantities?

Ex 1) $\underline{n=1}$ $\quad A = [\pm 1]$

identity or reflection

2) $\underline{n=2}$ $\quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotations

or $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection

or combinations

Def An $n \times n$ matrix U is orthogonal

if $$U(\underline{u}) \cdot U(\underline{v}) = \underline{u} \cdot \underline{v}$$

"U preserves dot product
(so also lengths, angles...)"

Exer TFAE = the following are equivalent

1) U is an orthog. matrix
2) cols of U form orthon. basis
3) $U^T = U^{-1}$

<u>Pf</u> 1) $\Rightarrow$ 2)

$$U = \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}$$

U orthog.

$$\underline{u}_i \cdot \underline{u}_j = U\underline{e}_i \cdot U\underline{e}_j = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

so cols are unit length and orthog. hence lin indep so a basis.

2) $\Rightarrow$ 3) Suppose $U = \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}$

with $\underline{u}_1, \dots, \underline{u}_n$ orthon. basis

$$U^T U = \begin{bmatrix} - & \underline{u}_1 & - \\ & \vdots & \\ - & \underline{u}_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}$$

$$= I_n \quad \text{since } \underline{u}_i \cdot \underline{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

so $U^T = U^{-1}$

<u>3) $\Rightarrow$ 1)</u> [<u>Aside</u> what does A^T really mean?

<u>Exer.</u> $\underline{u} \cdot A\underline{v} = A^T\underline{u} \cdot \underline{v}$

<u>Soln</u> $\underline{u}^T A \underline{v} = (A^T\underline{u})^T \underline{v}$

LHS RHS

Wow!

$(A^T\underline{u})^T = \underline{u}^T A$]

Back to 3) $\Rightarrow$ 1) Suppose $U^T = U^{-1}$

Calculate $U\underline{u} \cdot U\underline{v} = U^T U\underline{u} \cdot \underline{v}$

$$= \underline{u} \cdot \underline{v}$$

Done!

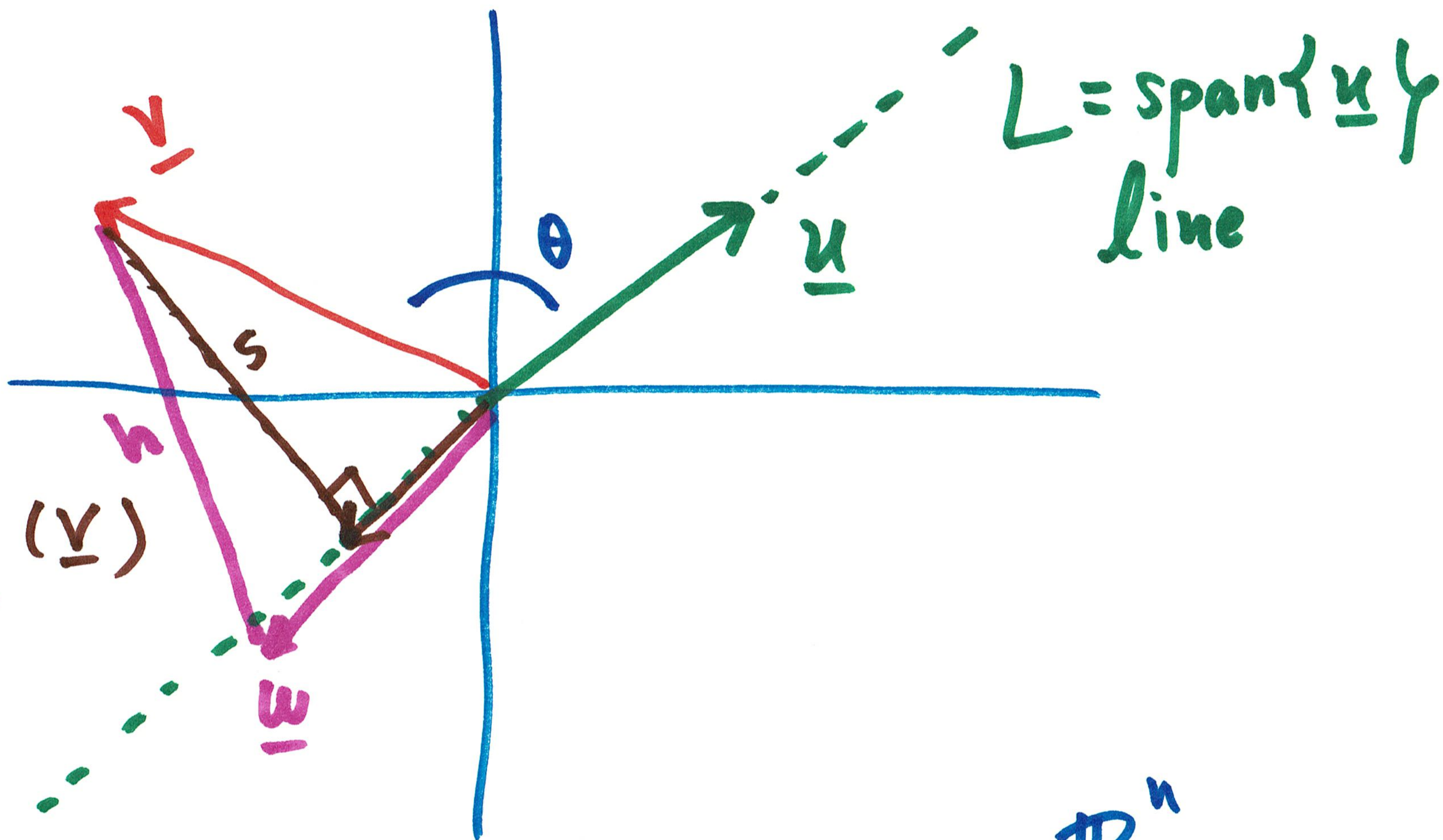
Recall from last time why we love orthog. bases $\underline{v}_1, \ldots, \underline{v}_n$

$$\underline{v} = a_1 \underline{v}_1 + \cdots + a_n \underline{v}_n \quad \text{where}$$

$$a_i = \frac{\underline{v} \cdot \underline{v}_i}{\underline{v}_i \cdot \underline{v}_i}$$

"coords have simple formula"

Geom. Interpretation of $\dfrac{\underline{v}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}$



$$\text{proj}_L(\underline{v}) = \frac{\underline{v}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\,\underline{u}$$

Remark When $\underline{u}$ is unit length

$$\mathrm{proj}_L(\underline{v}) = (\underline{v} \cdot \underline{u})\, \underline{u}$$
$$= \left(\|\underline{v}\| \cos\theta \right) \underline{u}$$

__Claim__ 1) $(\underline{v} - \text{proj}_L(\underline{v})) \perp \underline{u}$

__Why?__ $(\underline{v} - \text{proj}_L(\underline{v})) \cdot \underline{u}$

$$= \left(\underline{v} - \frac{\underline{v} \cdot \underline{u}}{\underline{u} \cdot \underline{u}}\,\underline{u}\right) \cdot \underline{u} = \underline{v} \cdot \underline{u} - \underline{v} \cdot \underline{u} = 0$$

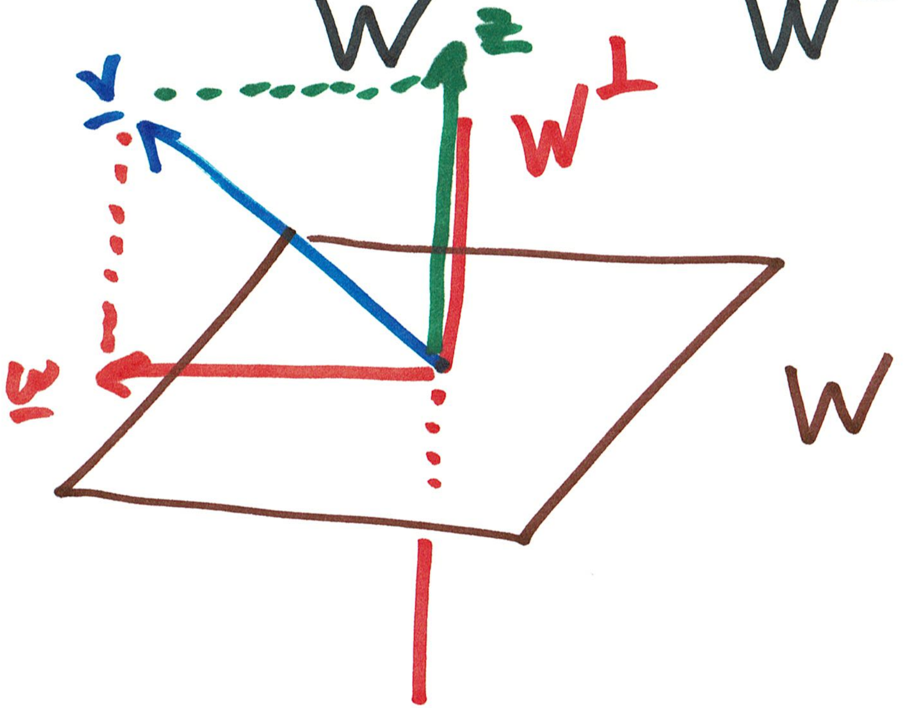
2) $\text{proj}_L(\underline{v})$ is __closest__ vector in L to $\underline{v}$

__Why?__ Pythag. Thm $h \geq s$

<u>Thm</u> $W \subset \mathbb{R}^n$ any subspace

Any vector $\underline{v} \in \mathbb{R}^n$ has unique decomp.

$$\underline{v} = \underset{\substack{\cap \\ W}}{\underline{w}} + \underset{\substack{\cap \\ W^\perp}}{z}$$



Def We call $\underline{w}$ from $\underline{v} = \underline{w} + \underline{z}$ the orthog proj of $\underline{v}$ onto W

$$\operatorname{proj}_W(\underline{v}) = \underline{w}$$

Rmk $\underline{z}$ will be orthog proj of $\underline{v}$ onto $W^{\perp}$

$$\operatorname{proj}_{W^{\perp}}(\underline{v}) = \underline{z}$$

<u>Pf</u> Suppose $\underline{w}_1, \ldots, \underline{w}_k$ is orthog basis of W

(We'll see in 10mins that orthog basis exists... Gram-Schmidt.)

Set $\underline{w} = \dfrac{\underline{v} \cdot \underline{w}_1}{\underline{w}_1 \cdot \underline{w}_1} \underline{w}_1 + \cdots + \dfrac{\underline{v} \cdot \underline{w}_k}{\underline{w}_k \cdot \underline{w}_k} \underline{w}_k$

Set $\underline{z} = \underline{v} - \underline{w}$

<u>Note</u> 1) $\underline{w} + \underline{z} = \underline{v}$ ☺ ✓

2) $\underline{w} \in W$ since $W = \text{Span}\{\underline{w}_1, \ldots, \underline{w}_k\}$

and $\underline{w}$ is lin comb of

$\underline{w}_1, \ldots, \underline{w}_k$ ✓

3) <u>Exer</u> check $\underline{z} \in W^{\perp}$

by showing $\underline{z} \cdot \underline{w}_i = 0$ ✓

all i

__Unique__? Suppose $\underline{v} = \underset{\in W}{\underline{w}'} + \underset{\in W^\perp}{\underline{z}'}$

So $\underline{w} + \underline{z} = \underline{w}' + \underline{z}'$

So $\underset{\in W}{\underline{w} - \underline{w}'} = \underset{\in W^\perp}{\underline{z}' - \underline{z}}$

But $W \cap W^\perp = \{\underline{0}\}$ since

$\underline{u} \in W \cap W^\perp$ satisfies $\underline{u} \cdot \underline{u} = 0$

so $\underline{u} = \underline{0}$

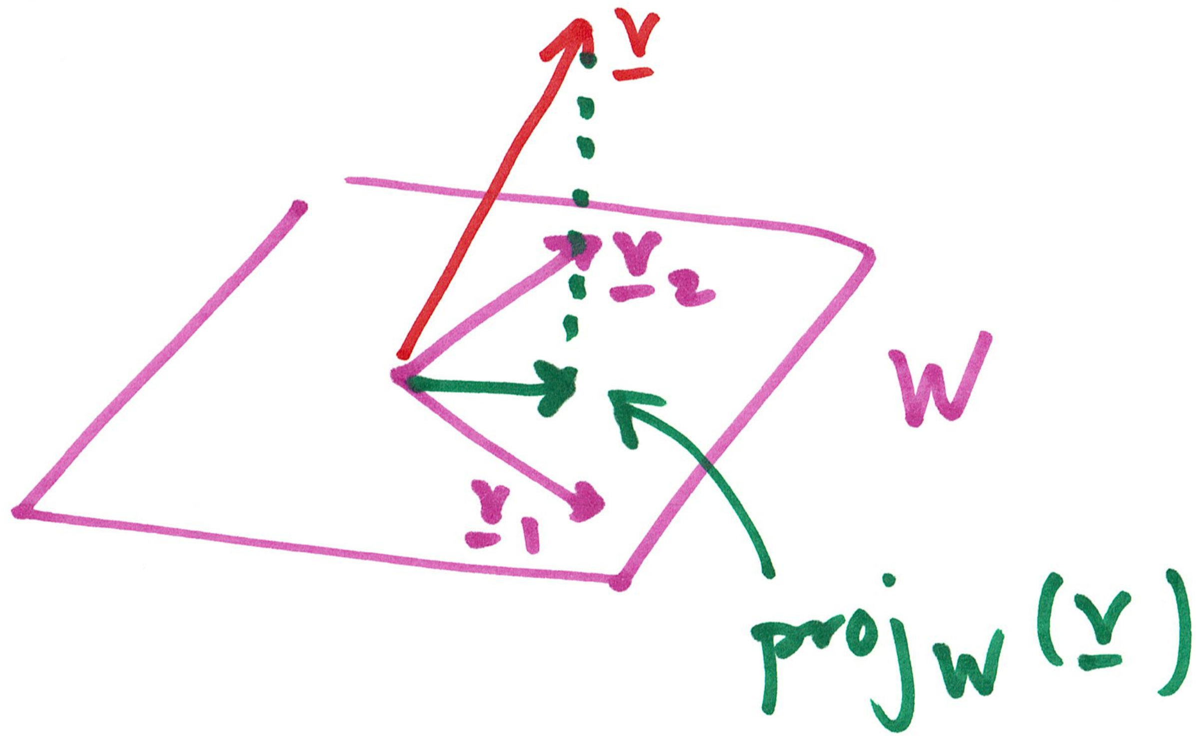
<u>Conclude</u> $\underline{w} - \underline{w}' = \underline{0} = \underline{z}' - \underline{z}$

So $\underline{w} = \underline{w}'$ and $\underline{z} = \underline{z}'$ $\blacksquare$

Yay!

Exer $W = \text{Span}\left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \subset \mathbb{R}^3$

Find $\text{proj}_W(\underline{v})$ where $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

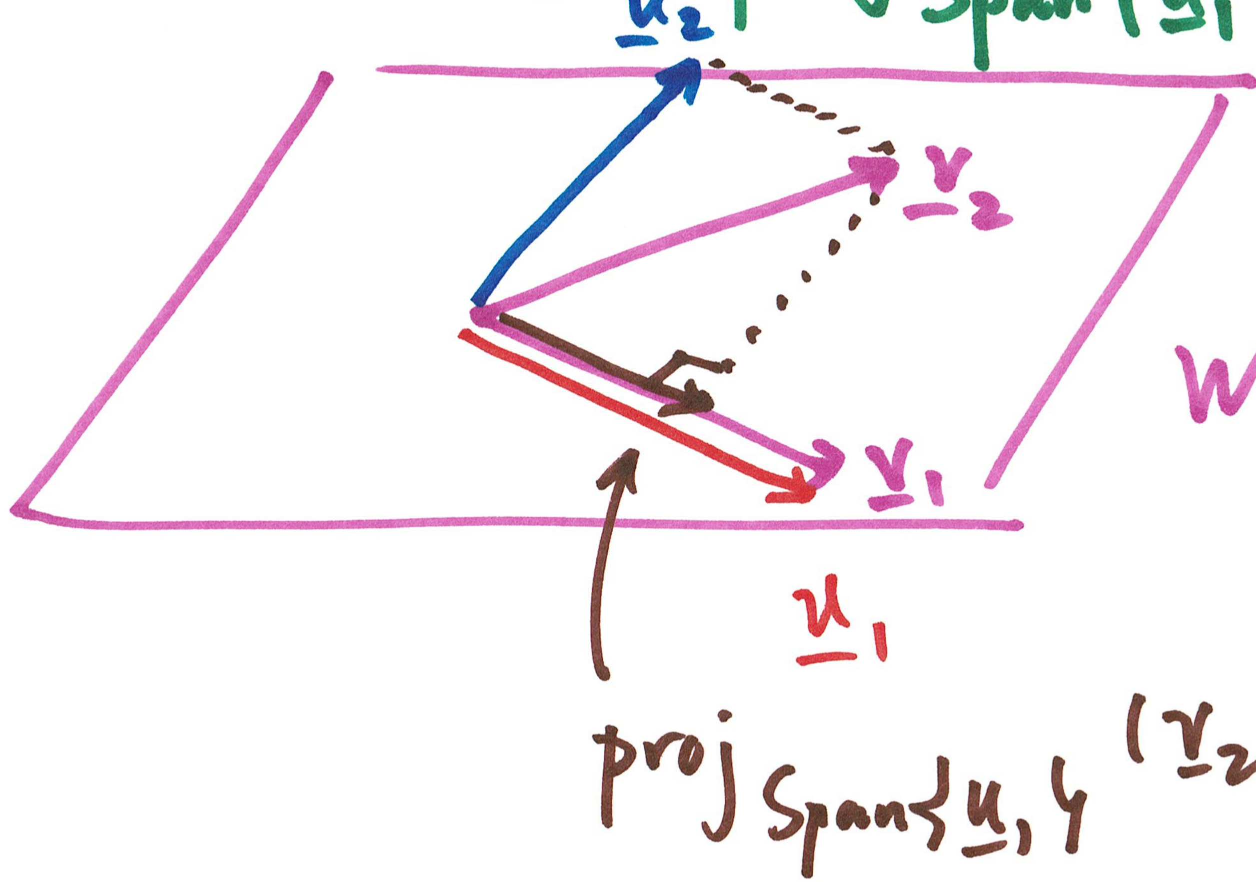


Sol'n Want orthog basis $\underline{u}_1, \underline{u}_2$ of W to apply formula $\text{proj}_W(\underline{v}) = \frac{\underline{v}\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\underline{u}_1 + \frac{\underline{v}\cdot\underline{u}_2}{\underline{u}_2\cdot\underline{u}_2}\underline{u}_2$

Method: Gram-Schmidt

Set $\underline{u}_1 = \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Take $\underline{u}_2 = \underline{v}_2 - \text{proj}_{\text{Span}\{\underline{u}_1\}}(\underline{v}_2)$



So $\underline{u}_2 = \underline{v}_2 - \dfrac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Check to be sure!

1) $\underline{u}_1 \cdot \underline{u}_2 = 0$

2) $\text{Span}\{\underline{u}_1, \underline{u}_2\} = W$

Now calc. $\text{proj}_W(\underline{v})$!

$$\text{proj}_W(\underline{v}) = \frac{\underline{v}\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\underline{u}_1 + \frac{\underline{v}\cdot\underline{u}_2}{\underline{u}_2\cdot\underline{u}_2}\underline{u}_2$$

$$\ldots = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

<u>Check</u> to <u>be sure</u>!

1) $\text{proj}_W(\underline{v}) \in W$

2) $\underline{v} - \text{proj}_W(\underline{v}) \in W^\perp$

Exer Find orthog basis for $W^{\perp}$

where $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^4$

Soln Find any basis

Note $W^{\perp} = \text{Null}\left(\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix} \right)$

(~~Rbrtem~~ Remember $\text{Row}(A)^{\perp} = \text{Null}(A)$!)

$\ldots \quad \underline{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

<u>Now</u> orthogo<u>nal</u>ize using G-S.

Set $\underline{u}_1 = \underline{v}_1$

Take $\underline{u}_2 = \underline{v}_2 - \mathrm{proj}_{\mathrm{span}\{\underline{u}_1\}}(\underline{v}_2)$

Take $\underline{u}_3 = \underline{v}_3 - \mathrm{proj}_{\mathrm{span}\{\underline{u}_1, \underline{u}_2\}}(\underline{v}_3)$

... $\underline{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$\underline{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{u}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1/5 \\ -2/5 \\ 1 \\ 0 \end{bmatrix} =$$ Simplify