Lecture 13

Applications of eigenvectors & eigenvalues

Wed Office Hours are back!  12-2pm, 891 Evans

Fri Quiz through §5.4

"Google can bring you back 100,000 answers. A librarian can bring you back the right one."

Neil Gaiman

Google's PageRank Algorithm

Problem rank pages that will
be most useful

Old School rank according to
appearance of keywords

__New idea__ rank pages by importance

__Importance__ is function of:

1) __popularity__: how many pages link to your page

2) __authority__: how important the pages linking to your page are

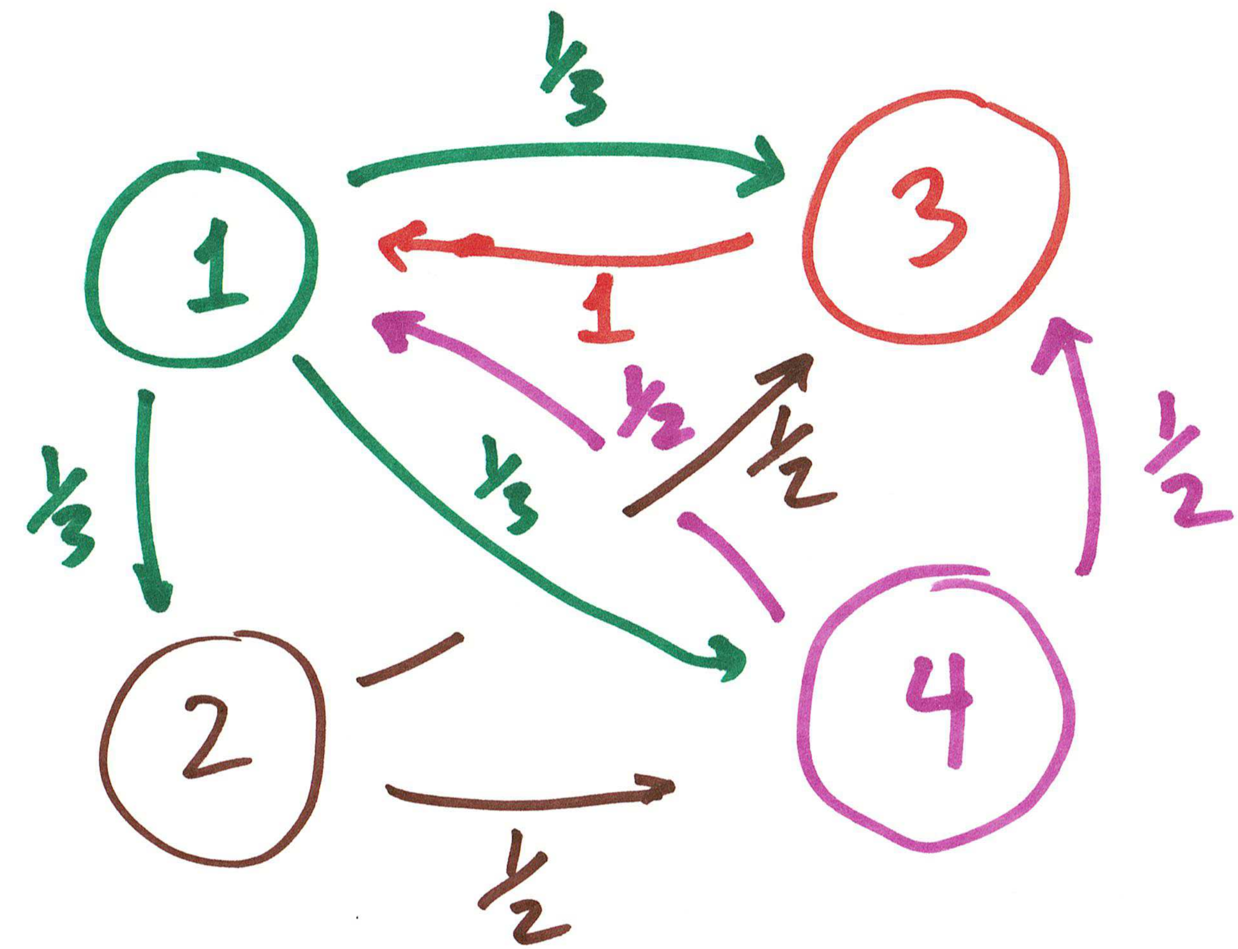
Some notation: Let's write $\underline{I}$ = internet = set of webpages $i \in I$

We'll write x_i = importance of website $i \in \underline{I}$

Importance vector $\underline{x} = [\, x_i \,]$, $i \in \underline{I}$.

# of components = size of I

__Example__ mini-internet $I = \{1, 2, 3, 4\}$



arrows are __weighted__ by $1 / \#$ of arrows out of a page

Organize weights into a matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

What eqn should the importance vector $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ satisfy?

Simplest possibility

$$A\underline{x} = \underline{x} \quad !$$

E-value/e-vector eqn for $\underline{\lambda = 1}$.

"Importance x_i is sum over all other pages of the importance x_j scaled by weight a_{ij}"

Find e-vector for A with e-value $\underline{\lambda = 1}$.

$E_1 = \text{Null}(A - 1 \cdot I)$

e-space for e-value $\lambda = 1$.

e-vectors are $\underline{v} \neq \underline{0} \in E_1$

... Soln $\underline{x} = \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$

Page rank

i	1	3	4	2
x_i	12	9	6	4

More theoretical application

$\dim V = n$
$\dim W = m$

Suppose $V \xrightarrow{T} W$ lin transf

Recall choise of bases β_V, β_W provide matrix

$$\boxed{A = T_{\beta_W}\, T\, T_{\beta_V}^{-1}}$$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "T_{\beta_V}"'] & W \arrow[d, "T_{\beta_W}"] \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m
\end{tikzcd}$$

More delicate situation $\dim V = n$

Suppose $V \xrightarrow{T} V$ li transf.

How "nice" can we make matrix of T by choosing single basis β of V?

$$\boxed{A = T_\beta T T_\beta^{-1}}$$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "T_\beta"'] & V \arrow[d, "T_\beta"] \\
\mathbb{R}^n \arrow[r, "A"'] & \mathbb{R}^n
\end{tikzcd}$$

__Thm__ Suppose $\beta = \{\underline{v}_1, \ldots \underline{v}_n\}$ is a basis of e-vectors for T

$$T\underline{v}_i = \lambda_i \underline{v}_i$$

Then matrix A of T wrt β is __diagonal__

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

"As nice as possible"

<u>Pf</u>. Matrix A is given by

$$A = \begin{bmatrix} | & & | \\ T_\beta T T_\beta^{-1}(\underline{e}_1) & \cdots & T_\beta T T_\beta^{-1}(\underline{e}_n) \\ | & & | \end{bmatrix}$$

Now calc. ith col:

$$T_\beta T T_\beta^{-1}(\underline{e}_i) = T_\beta T(\underline{v}_i)$$

$$= T_\beta(\lambda_i \underline{v}_i) = \lambda_i T_\beta(\underline{v}_i)$$

$$= \lambda_i \underline{e}_i$$

So conclude

$$A = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & \lambda_n \end{bmatrix} \qquad \blacksquare$$

<u>Def</u> $\dim V = n$

A lin transf $T : V \to V$ is <u>diagonalizable</u> if there is a basis $\beta = \{\underline{v}_1, \ldots, \underline{v}_n\}$ so that matrix A of T wrt β is diagonl

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

Previous Thm says

If there is basis of e-vectors for T
then T is diagonalizable

Conversely, also have

Thm $\dim V = n$
Suppose $T: V \to V$ is diagonalizable
then there is a basis
$\beta = \{\underline{v}_1, \ldots, \underline{v}_n\}$ of e-vectors.

Exer Prove the converse thm!

Caution Not all lin transf $T : V \to V$ are diagonalizable!

Ex: $V = \mathbb{R}^2$, T given by matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Let's see why T is <u>not</u> diagonalizable

Check there is <u>not</u> a basis of e-vectors.

<u>Step</u> (1) Find e-values

$$\chi_A(t) = (-t)^2 = t^2$$

<u>roots</u>: ~~are~~ $\lambda = 0$ with mult 2.

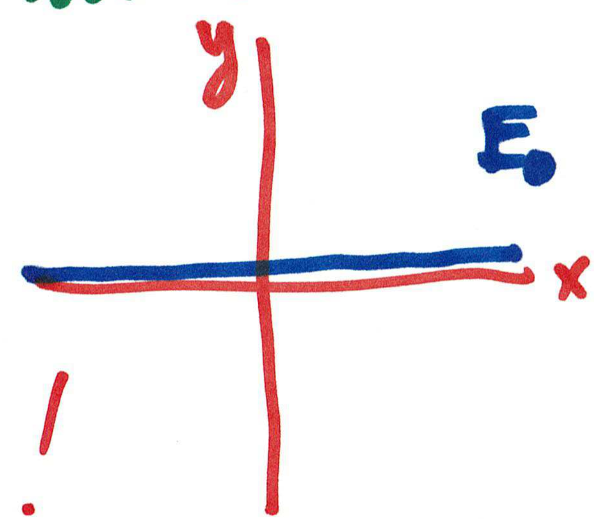
Step 2 dim E_0 is either 1 or 2.

$$E_0 = \text{Null}(A - 0 \cdot I) = \text{Null}(A)$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

e-vector

All other e-vectors are non-zero scales of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$



So there is <u>not</u> a basis of e-vectors!

More examples

1) $V = \mathbb{R}^2$, $T : V \to V$ given by matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$ Diagonalizable?

Step (1) $\chi_A(t) = (1-t)(2-t)$

$\lambda = 1, 2$ e-values

Step (2) $\dim E_1 = \dim E_2 = \underline{1}$.

Find <u>e-vectors</u>:

$$E_1 = \mathrm{Null}(A - 1\cdot I)$$

$$= \mathrm{Null}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \mathrm{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= \underline{v}_1} \right\}$$

$$E_2 = \mathrm{Null}(A - 2I)$$

$$= \mathrm{Null}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{= \underline{v}_2} \right\}$$

Take basis $\beta = \{ \underline{v}_1, \underline{v}_2 \}$

2) $V = \mathbb{R}^3$, $T : V \to V$ given by matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 4 & 2 \end{bmatrix}$$ Diagonalizable?

<u>Step</u> (1) $\chi_A(t) = (2-t)(1-t)^2$

$\lambda = 1$, $\lambda = 2$

mult $= 2$ mult $= 1$

Step (2) Want $\sum_{\substack{\lambda \\ \text{e-values}}} \dim E_\lambda = \dim V$

$\underline{\lambda = 2}$ $\dim E_2 = 1$ ... solve for e-vector $\underline{v}_1$

$\underline{\lambda = 1}$ $\dim E_1 = 1$ or 2

Not diagonalizable

Calc $E_1 = \text{Null}(A - 1 \cdot I) = \ldots$

$\xrightarrow{\text{REF}} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

only 1 free var so $\dim E_1 = 1$

<u>Summary of strategy</u> to check if $T: V \to V$ is <u>diagonalizable</u>.

Equivalently check if basis of e-vectors

<u>Step</u> 1 Find roots of $\chi_A(t)$ — next time we'll see complex roots factor $\chi_A(t)$ always

<u>Step</u> 2 Find e-spaces

If $\dim V = \sum_{\lambda \text{ e-value}} \dim E_\lambda$ then diagonalizable! Else, not.