Now the fun stuff! **Lecture** 12

Eigen*vectors* and Eigen*values*

Or *How to Make Billions with Lin. Alg.*

Fri Quiz through §4.6

"One person's craziness is another person's reality."

Tim Burton

We are world's experts in solving lin systs

$$A\,\underline{x} = \underline{b}$$

m×n matrix

n-vector of vars

n-vector of values

A, $\underline{b}$ are given

we seek $\underline{x}$

Also makes sense more abstractly

$$T\,\underline{x} = \underline{b}$$

lin transf

vector in V

vector in W

$$T : V \to W$$

Second great eqn of Lin Alg

number

$$A \, \underline{x} = \lambda \, \underline{x}$$

$n \times n$ matrix

square!

$\square$

n-vector of vars

A given

we seek $\underline{x}$, λ

Also makes sense more abstractly

number

$$T \underline{x} = \lambda \underline{x}$$

$T : V \to V$

lin transf

vector in V

Note $\underline{x} = \underline{0}$ always solves $A\underline{x} = \lambda \underline{x}$
for *any* λ

Agreement We'll never ever
ever consider $\underline{x} = \underline{0}$
as an interesting soln.

Terminology: Suppose λ, $\underline{x} \neq \underline{0}$ solve $A\underline{x} = \lambda \underline{x}$

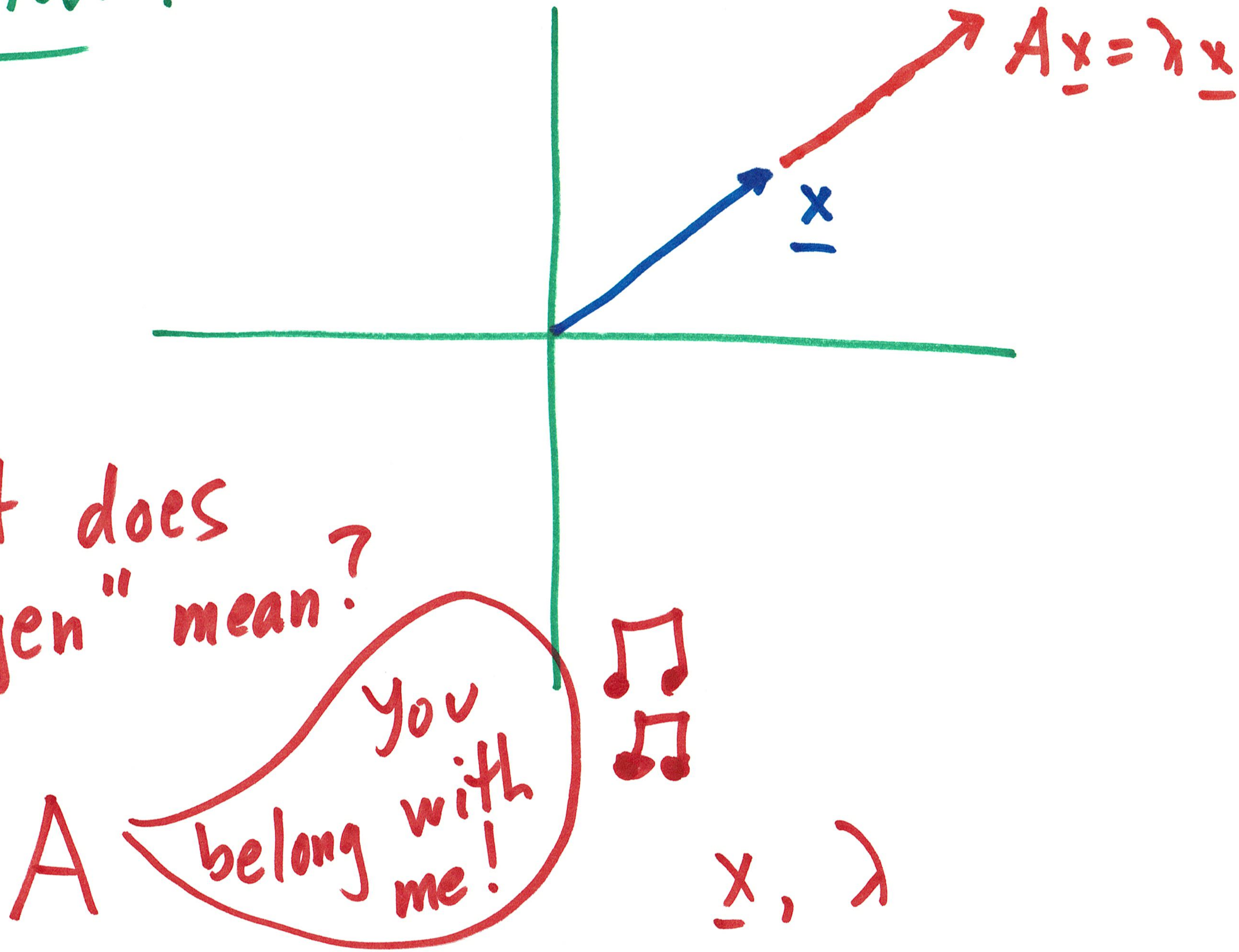
Then we say:

λ is an **eigenvalue** (e-value) for A

$\underline{x}$ is an **eigenvector** (e-vector) for A

Cartoon:

What does "eigen" mean?



__Note__ 1) If $\underline{x}$ is an e-vector so $A\underline{x} = \lambda \underline{x}$ then λ is __unique__!

__Why?__ $\lambda \underline{x} = A\underline{x} = \mu \underline{x}$

$\Longrightarrow \lambda \underline{x} - \mu \underline{x} = \underline{0}$

$\Longrightarrow (\lambda - \mu)\underline{x} = \underline{0}$, $\underline{x} \neq \underline{0}$

$\Longrightarrow \lambda = \mu$

2) Suppose λ is e-value ~~with~~ for ~~fansame~~ some e-vector $\underline{x}$

so $A\underline{x} = \lambda \underline{x}$

<u>Caution</u> $\underline{x}$ is <u>not nec</u> unique!

<u>Ex</u> $A = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

Any $\underline{x} \neq \underline{0} \in \mathbb{R}^2$ is an e-vector for $\lambda = 14$

__Exer__ Find e-values & e-vectors for

$$A = \begin{bmatrix} 7 & 0 & \cdots & & & 0 \\ 0 & 0 & & & & \vdots \\ \vdots & & -1 & & & \vdots \\ \vdots & & & 0 & & 0 \\ & & & & -1 & 0 \\ 0 & \cdots & & & 0 & 3 \end{bmatrix}$$ __diag.__ __matrix__

__Soln__

$\underline{\lambda = 7}$: $\underline{x} = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $a \neq 0$

$\underline{\lambda = 0}$: $\underline{x} = \begin{bmatrix} 0 \\ a \\ 0 \\ b \\ 0 \\ 0 \end{bmatrix}$, $a \neq 0$ or $b \neq 0$

$\underline{\lambda = -1}$: $\underline{x} = \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ b \\ 0 \end{bmatrix}$, $a \neq 0$ or $b \neq 0$

$\underline{\lambda = 3}$: $\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \end{bmatrix}$, $a \neq 0$

How do we know we found all e-values & e-vectors?

Need some science...

General strategy for solving $A\underline{x} = \lambda \underline{x}$

Special case $\lambda \underline{\underline{=}} 0$ $\quad A\underline{x} = \underline{0}$!

Old friend ... homog lin syst.

$\{$e-vector for $\lambda = 0\} = \text{Null}(A) \setminus \{\underline{0}\}$

What about $\lambda \neq 0$?

<u>General case</u> <u>any λ</u>

$$A\underline{x} = \lambda \underline{x} \iff A\underline{x} - \lambda\underline{x} = \underline{0}$$

$$\iff (A - \lambda \underline{I})\,\underline{x} = \underline{0}$$

$$\lambda \underline{I} = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}$$

scales vectors by λ.

Our old friend again ... homog lin syst for coeff matrix $(A - \lambda \underline{I})$

Terminology If λ is an e-value of A

we call

$$E_\lambda = \text{Null}\,(A - \lambda I)$$

the **eigenspace (e-space)** of λ

$$\text{Null}(A - \lambda I) = \{\text{e-vectors for } \lambda\}$$

$\setminus \{\underline{0}\}$

$\triangle$ 2 $\underline{0}$ is never an e-vector

Key question when is λ an e-value of A?

Equivalently when is

$$\dim E_\lambda = \dim \mathrm{Null}(A - \lambda I) > 0 \, ?$$

Beautiful Criterion $\dim E_\lambda = \dim \mathrm{Null}(A-\lambda I) > 0$

$\iff E_\lambda = \mathrm{Null}(A-\lambda I) \neq \{\underline{0}\}$

$\iff \det(A-\lambda I) = 0$

$\iff \lambda$ is a root of **characteristic polynomial**

$$\chi_A(t) = \det(A - tI)$$

Recall λ is a root of char poly $\chi_A(t)$ means λ satisfies char eqn

$$\chi_A(t) = 0$$

i.e. $\boxed{\chi_A(\lambda) = 0}$

<u>Summary of strategy</u> for solving $A\underline{x} = \lambda\underline{x}$

1) <u>Find e-values</u> find <u>roots</u> λ of char poly $\chi_A(t) = \det(A - tI)$

2) <u>Find e-spaces</u> find $E_\lambda = \text{Null}(A - \lambda I)$ for each e-value λ

e-<u>ve</u>ctors $\underline{x} \neq \underline{0} \in E_\lambda$

<u>Exer</u> Find e-values λ and e-spaces E_λ for each A

1) $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ <u>Step</u> 1) $\chi_A(t) =$

$= \det\left(\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \right)$

$= \det\left(\begin{bmatrix} 2-t & 3 \\ 0 & -1-t \end{bmatrix} \right) = (2-t)(-1-t) - 0 \cdot 3$

Good times already factored.

∴ roots are $\underline{\lambda = 2}$ $\qquad$ $\underline{\lambda = -1}$

Step(2)

$(A - 2I) = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}$ $\qquad$ $(A - (-1)I) = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$

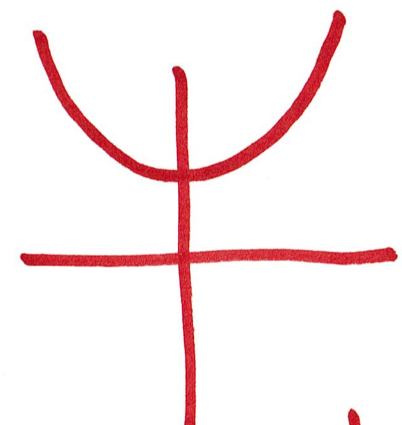
$E_2 = \text{Null}(A - 2I)$ $\qquad$ $E_{-1} = \text{Null}(A - (-1)I)$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ $\qquad$ $= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Step (1) $\chi_A(t)$

$= \det(A - tI)$

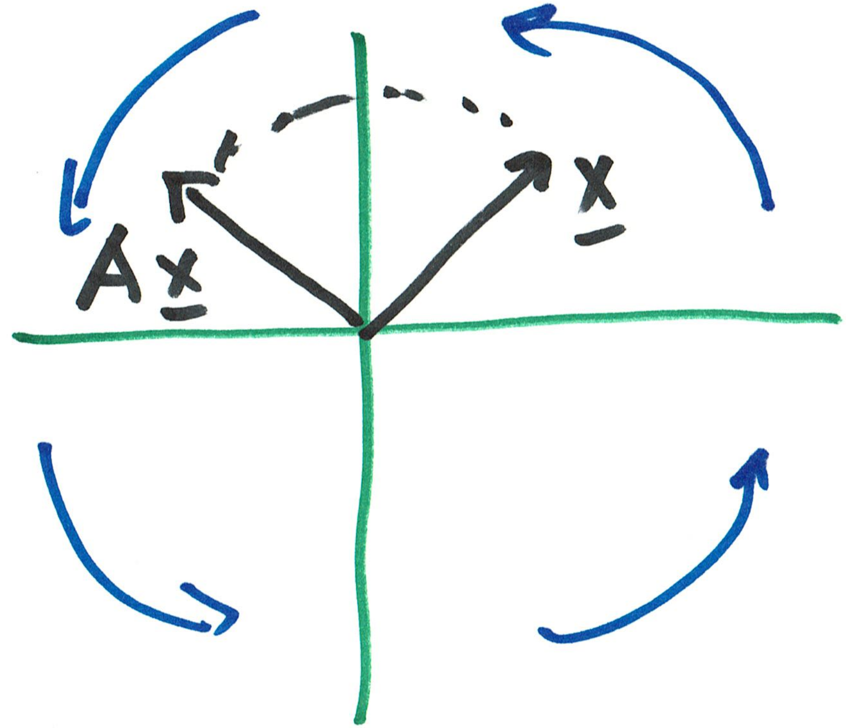
$= \det\left(\begin{bmatrix} -t & -1 \\ 1 & -t \end{bmatrix}\right) = t^2 + 1$



No $\underline{\underline{\text{real}}}$ e-values or e-vectors for A $\Leftarrow$ no $\underline{\underline{\text{real}}}$ roots

Later... $\underline{\underline{\text{complex}}}$ roots

<u>Geometrically</u> A rotates by $\frac{\pi}{2}$



Only $\underline{x} = \underline{0}$ is rotated to scale of itself

3) $A = \begin{bmatrix} 3 & 0 \\ -7 & 3 \end{bmatrix}$ <u>Step</u> (1) $\chi_A(t) =$

$= \det(A - tI) =$

$= \det\left(\begin{bmatrix} 3-t & 0 \\ -7 & 3-t \end{bmatrix}\right) = (3-t)^2$

only e-value $\underline{\lambda = 3}$

<u>Step</u> (2) $E_3 = \text{Null}(A - 3I)$

$= \text{Null}\left(\begin{bmatrix} 0 & 0 \\ -7 & 0 \end{bmatrix}\right)$

$E_3 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\dim E_3 = 1$

3') $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ same char poly

$\chi_A(t) = (3-t)^2$

only e-value $\underline{\lambda = 3}$

$E_3 = \mathbb{R}^2$ $\dim E_3 = 2$

<u>Fact</u> If λ e-value of A,

then

$1 \leq \dim E_\lambda \leq$ mult. of λ as root of char poly $\chi_A(t)$

Exer Suppose A is 3×3 with

$$\chi_A(t) = (7-t)^3$$

Exhibit matrices A with $\dim E_7$ each of $1, 2, 3$

Soln $\underline{\dim E_7 = 3}$ $\quad A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ unique poss.

$\underline{\dim E_7 = 2}$ $\quad A = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ $\cdots$ many other poss.

$\underline{\dim E_7 = 1}$ $\quad A = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$ $\cdots$