Lecture 10 Dimension & Rank!

Fri: Quiz through §4.4

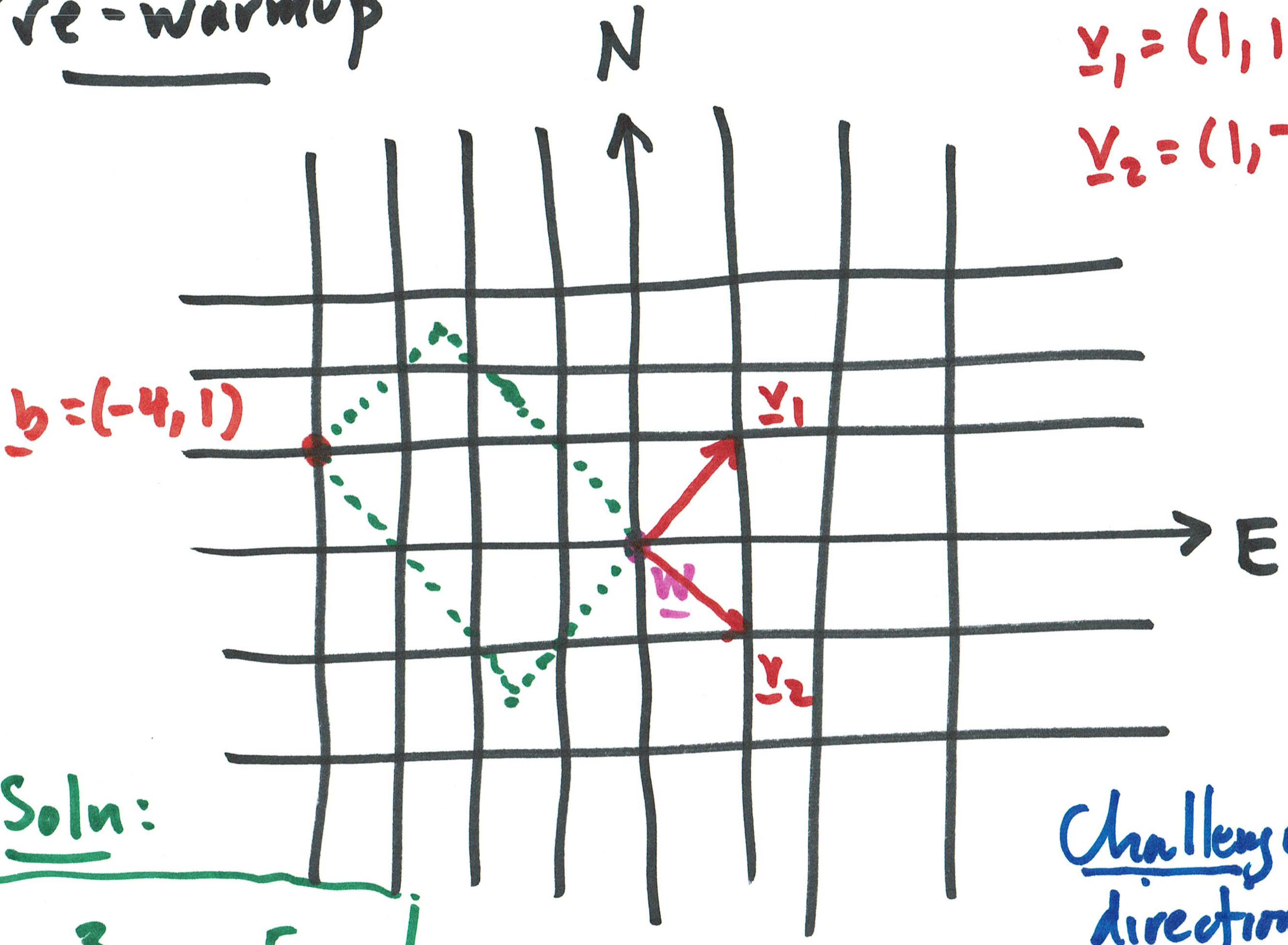
Next week: Business as usual

"There is a fifth dimension beyond
that which is known to man...
This is the dimension of imagination."
Rod Serling

Pre-warmup

$\underline{v}_1 = (1,1)$ NE

$\underline{v}_2 = (1,-1)$ SE



$\underline{w} = (0,0)$

Challenge: directions from $\underline{w}$ to $\underline{b}$ in basis $\underline{v}_1, \underline{v}_2$.

Soln:

$$\boxed{\underline{b} = -\frac{3}{2}\underline{v}_1 - \frac{5}{2}\underline{v}_2}$$

coords of $\underline{b}$ wrt basis $\underline{v}_1, \underline{v}_2$

<u>Warmup</u> Consider $T: \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

1) Find basis for $\underset{\text{Col}(A)}{\underset{\shortparallel}{\text{Image}(T)}}$, $\text{Null}(T)$

2) Find coords wrt your base for

$$\underline{v} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \in \text{Null}(T), \quad \underline{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in \text{Image}(T)$$

$$\underline{\text{Soln}}\ 1)\ A \rightsquigarrow A_{ref} = \begin{array}{cccc} \left[\begin{array}{cccc} \textcircled{1} & 1 & 0 & 1 \\ 0 & \textcircled{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \end{array} \\ \begin{array}{cc} \text{pivot cols} & \text{free cols} \end{array} \end{array}$$

$\underline{\text{Basis for Image}(T)}$ $\underline{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underline{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

cols of $\underset{\sim\sim\sim\sim}{\text{original}}$ matrix

corr to pivot cols of REF

$\underline{\text{Basis for Null}(T)}$: set $x_3 = 1, x_4 = 0$ free vars

solve: $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ for pivot vars set $x_3 = 0, x_4 = 1$ free vars

solve: $\underline{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ for pivot vars

2) Solve: $\underline{v} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 2 \end{bmatrix} = a_1 \underline{v}_1 + a_2 \underline{v}_2$

$= 3\underline{v}_1 + 2\underline{v}_2$

Coords of $\underline{v}$: $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

free var

free var

How did we arrive at $\underline{v}_1, \underline{v}_2$ as basis for Null(T)?

Any soln of $A\underline{x} = \underline{0}$ is of form:

- allow free vars x_3, x_4 to be any numbers
- then solve for pivot vars x_1, x_2

$$\underline{x} = \begin{bmatrix} x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Solve $\underline{w} = c_1 \, \underline{w}_1 + c_2 \, \underline{w}_2$

$$\underline{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0 \cdot \underline{w}_1 + 1 \cdot \underline{w}_2$$

since $\underline{w} = \underline{w}_2$

Coords of $\underline{w}$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Are bases unique No! But...

Thm If vect sp V has a basis $\beta = \{ \underline{v}_1, \dots, \underline{v}_k \}$ with k elements then any other basis $\beta' = \{ \underline{w}_1, \dots, \underline{w}_\ell \}$ will have the same number of elements $\ell = k$

Pf Consider coord map

$$T_\beta : V \longrightarrow \mathbb{R}^k$$

$$T_\beta(\underline{v}) = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$ determined by $\underline{v} = a_1\underline{v}_1 + \dots + a_k\underline{v}_k$

This map T_β is inj and surj.

Inverse lin transf $T_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}\right) = a\underline{v}_1 + \dots + a_k\underline{v}_k$

Apply T_β to basis β' to obtain

$$T_\beta(\underline{w}_1), \ldots, T_\beta(\underline{w}_\ell) \in \mathbb{R}^k$$

Claim $T_\beta(\underline{w}_1), \ldots, T_\beta(\underline{w}_\ell)$ are lin indep and span $\mathbb{R}^k$

Why for span: take some $\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \in \mathbb{R}^k$

Apply T_β^{-1} to obtain $\underline{w} = T_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}\right) \in V$

Since $\underline{w}_1, \dots, \underline{w}_\ell$ are a basis of V they span so we can write

$$T_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}\right) = \underline{w} = c_1 \underline{w}_1 + \dots + c_\ell \underline{w}_\ell$$

for some numbers $c_1, \dots, c_\ell$

Apply T_β to return to $\mathbb{R}^k$:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = T_\beta(\underline{w}) = c_1 T_\beta(\underline{w}_1) + \dots + c_\ell T_\beta(\underline{w}_\ell)$$

So indeed $T_\beta(\underline{w}_1), \ldots, T_\beta(\underline{w}_\ell)$
span $\mathbb{R}^k$!

Exer Show also lin indep.

Conclusion $\underline{\ell \leq k}$ since $T_\beta(\underline{w}_1), \ldots, T_\beta(\underline{w}_\ell)$
lin indep

$\underline{\ell \geq k}$ since $T_\beta(\underline{w}_1), \ldots, T_\beta(\underline{w}_\ell)$
span $\mathbb{R}^k$

$\blacksquare$

Def dimension of vect sp $\underline{V}$

$$\dim V = \begin{cases} k & \text{if } V \text{ has basis } \beta \text{ with } k \text{ elts} \\ \underset{\text{" infinity!}}{\infty} & \text{else} \end{cases}$$

Justified (unambiguous)
thanks to Thm.

Examples

1) $V = \mathbb{R}^n$ dim $= n$

2) $V = P_n = \{$poly fns of deg $\leq n\}$ dim $= n+1$

std basis $p_0(x) = 1$, $p_1(x) = x, \dots,$

$p_n(x) = x^n$

3) $V = S_0 = \{(a_1, a_2, a_3, \dots)$ eventually always $0\}$

dim $= \infty$

std basis $\underline{e}_1 = (1,0,0,\dots)$, $\underline{e}_2 = (0,1,0,0,\dots)$

$\dots$

Def. Let $T:V \to W$ lin transf.

nullity of T $= \dim \text{Null}(T)$

rank of T $= \dim \text{Image}(T)$

<u>Exer</u> Calc. nullity and rank of $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$A = \begin{bmatrix} t & t & t \\ -t & -t^2 & -t^3 \end{bmatrix}$$

as functions of t!

Soln $A \rightsquigarrow A_{ref} = \begin{bmatrix} \underset{\sim}{t} & t & t \\ 0 & \underset{\sim}{t-t^2} & t-t^3 \end{bmatrix}$

	nullity	rank
t=0	3	0
t=1	2	1
t≠0,1	1	2

Is nullity + rank = dim domain
a coincidence?

Rank Thm Let $T : V \to W$ lin transf
dim V finite

Then dim V = nullity + rank!

Proof in special case $V = \mathbb{R}^n$, $W = \mathbb{R}^m$

T is given by matrix $\underset{m \times n}{A}$

$$\begin{aligned} \dim V = n &= \#\text{ of cols} \\ &= \#\text{ of free cols} \\ &\quad + \\ &\quad \#\text{ of pivot} \\ &= \text{nullity} + \text{rank}\ ! \end{aligned}$$

▨