You have 20 minutes to complete this quiz. To receive full credit, you must justify your answers.

Name : ______________________________

1. (5 points) Find an invertible matrix P such that $A = PDP^{-1}$, where

$$A = \begin{bmatrix} -2 & 0 & 2 \\ -8 & 2 & 4 \\ -4 & 0 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution: We are told that A is similar to D, so we know A's eigenvalues are 2 and 0. To find a basis of eigenvectors of A, first we find bases for $\text{Nul}(A-2I)$ and $\text{Nul}(A-0I) = \text{Nul}(A)$. Starting with $\text{Nul}(A-2I)$,

$$\text{Nul}(A-2I) = \text{Nul}\begin{bmatrix} -4 & 0 & 2 \\ -8 & 0 & 4 \\ -4 & 0 & 2 \end{bmatrix} = \text{Nul}\begin{bmatrix} -4 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul}\begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so for $\mathbf{x}$ to satisfy $(A-2I)\mathbf{x} = \mathbf{0}$, we need $2x_1 = x_3$, with x_2 and x_3 free. Hence $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\text{Nul}(A-2I)$.

Now computing $\text{Nul}(A-0I)$,

$$\text{Nul}(A-0I) = \text{Nul}\begin{bmatrix} -2 & 0 & 2 \\ -8 & 2 & 4 \\ -4 & 0 & 4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_1 = x_3$ and $x_2 = 2x_3$, with x_3 free. Thus $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Nul}(A-0I)$.

Putting the eigenvectors of A in the same order as the corresponding eigenvalues listed in D, we get that

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

satisfies $A = PDP^{-1}$.

2. (5 points) Determine if the following set of vectors is orthogonal:

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \right\}$$

Solution: We must check that for every pair of vectors (say $\mathbf{u}$ and $\mathbf{v}$) in the set, we have $\mathbf{u} \cdot \mathbf{v} = 0$. Well,

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = 2 + 8 - 10 = 0,$$

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = 8 - 8 + 0 = 0,$$

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = 4 - 4 + 0 = 0.$$

So indeed the vectors are orthogonal.