You have 20 minutes to complete this quiz. To receive full credit, you must justify your answers.

Name : ______________________________

1. (5 points) Find all real eigenvalues of the following matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 4 & 9 & 2 \end{bmatrix}$$

Solution: We need to solve $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ 4 & 9 & 2-\lambda \end{bmatrix}$$

Cofactor expanding along the first row, we get

$$(1-\lambda)((2-\lambda)^2 - 9) = 0$$

So $\lambda = 1$ or

$$\begin{aligned} (2-\lambda)^2 - 9 &= 0 \\ (2-\lambda)^2 &= 9 \\ 2-\lambda &= \pm 3 \\ \lambda &= -1, 5 \end{aligned}$$

Thus, the real eigenvalues are $\lambda = 1, -1, 5$.

2. (5 points) Find the $\mathcal{B}$-matrix of the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ which satisfies

$$T(1 - 3t^2 + 2) = 2 - 6t^2 + 4$$
$$T(2t + t^2) = 6t + 3t^2$$
$$T(1 + t) = 0.$$

where $\mathcal{B}$ is the basis $\{1 - 3t^2 + 2, 2t + t^2, 1 + t\}$ of $\mathbb{P}_2$.

Solution: Let $\mathbf{b}_1 = 1 - 3t^2 + 2$, $\mathbf{b}_2 = 2t + t^2$, and $\mathbf{b}_3 = 1 + t$, so that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Notice that

$$T(\mathbf{b}_1) = 2\mathbf{b}_1, T(\mathbf{b}_2) = 3\mathbf{b}_2, T(\mathbf{b}_1) = 0\mathbf{b}_1,$$

so that

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, [T(\mathbf{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, [T(\mathbf{b}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence the $\mathcal{B}$-matrix of T is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(Notice that this matrix is diagonal; that's just because $\mathcal{B}$ was a basis of eigenvectors of T).