You have 20 minutes to complete this quiz. To receive full credit, you must justify your answers.

Name : ____________________

1. (5 points) Determine if the set $D = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is once differentiable } \}$ of once differentiable functions is a subspace of the vector space $V = \{f : \mathbb{R} \to \mathbb{R}\}$ of all functions on $\mathbb{R}$.

Solution: Since differentiable functions are functions, $D \subseteq V$. Then we need to check three things to determine if D is a subspace of V. First, we must check that the zero vector is in D; second, we must check that D is closed under vector addition; and third, we must check that D is closed under multiplication by scalars.

The zero vector in V is the zero function $f(x) = 0$, which is certainly differentiable. Thus, $\mathbf{0} \in D$.

Next, suppose that $f, g \in D$; i.e., that f and g are two differentiable functions. Our knowledge of calculus I tells us that if f and g are differentiable, so is $f + g$. Hence, $f + g \in D$.

Finally, let $f \in D$ and let $c \in \mathbb{R}$. Again, calculus I tells us that if f is differentiable and c is a constant, so is cf.

Thus, all three properties are satisfied, and D is a subspace of V.

2. (5 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 10 \\ -3 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute the volume of $T(B)$, where B is the box $-1 \le x_1 \le 2$, $0 \le x_2 \le 1$, $1 \le x_3 \le 3$.

Solution: If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 10 \\ -3 & -3 & 1 \end{bmatrix}$, the volume of $T(B)$ is equal to $|\det(A)| \cdot \mathrm{Vol}(B)$.

B is a parallelepiped with sides parallel to the axis, so its volume is

$$\mathrm{Vol}(B) = (2 - (-1)) \cdot (1 - 0) \cdot (3 - 1) = 6.$$

Next we compute the determinant of A. If we subtract 2 times row 1 from row 2 and add 3 times row 1 to row 3, we do not change the determinant of A, so we may instead compute the determinant of $A' = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 0 \\ 0 & 6 & 16 \end{bmatrix}$.

Using the cofactor expansion in the first column, we see that

$$\det(A') = 1 \cdot \det\left(\begin{bmatrix} 2 & 0 \\ 6 & 16 \end{bmatrix}\right) - 0 \cdot \det\left(\begin{bmatrix} 3 & 5 \\ 6 & 16 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}\right) = 1 \cdot 2 \cdot 16 - 0 + 0 = 32.$$

Thus, the volume of $T(B)$ is

$$\mathrm{Vol}(T(B)) = |\det(A)| \cdot \mathrm{Vol}(B) = |\det(A')| \cdot 6 = 32 \cdot 6 = 192.$$