You have 20 minutes to complete this quiz. To receive full credit, you must justify your answers.

Name : ______________________________

1. (5 points) Find the general solution to the following system.

$$y_1' = y_1 - 3y_2$$
$$y_2' = 2y_1 - 4y_2$$

Solution: We wish to solve

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic polynomial of the matrix is $(1-\lambda)(-4-\lambda)+6 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$. Thus, the eigenvalues are -1 and -2. By row reducing, we find that the corresponding eigenvectors are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

Thus, the general solution is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = C_1 e^{-x} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 e^{-2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2. (5 points) Write down the sine series of the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = x$.

Solution: We compute the coefficients as follows.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \\ &= -\frac{1}{n\pi}\int_{-\pi}^{\pi} x \, d(\cos nx) \\ &= -\left[\frac{x\cos nx}{n\pi}\right]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} \cos nx \, dx \\ &= -\frac{\pi\cos n\pi - (-\pi)\cos -n\pi)}{n\pi} + \frac{\sin n\pi - \sin -n\pi}{n^2\pi} \\ &= -\frac{2(-1)^n}{n} \end{aligned}$$

Thus, the sine series is

$$-2\sum_{n=1}^{n=\infty} \frac{(-1)^n \sin nx}{n}.$$