You have 20 minutes to complete this quiz. To receive full credit, you must justify your answers.

Name : ______________________________

1. (5 points) Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}.$$

Solution: Recall that least-squares solutions of the inconsistent system $A\mathbf{x} = \mathbf{b}$ satisfy the equation $A^T A\mathbf{x} = A^T\mathbf{b}$. We compute those quantities.

$$A^T A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

Thus, we get that $3x_1 = 9$ and $24x_2 = 1/2$ so the least-squares solution is

$$\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}.$$

2. (5 points) Orthogonally diagonalize (Find an orthogonal matrix P and diagonal matrix D so that $A = PDP^{-1}$) the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

which has eigenvalues $\lambda = 2, 5$.

Solution: We start by finding bases of the eigenspaces.

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis of } E_2$$

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis of } E_5$$

Note that the eigenvector for $\lambda = 5$ is orthogonal to the eigenvectors for $\lambda = 2$ (as will always be true), but the two eigenvectors for $\lambda = 2$ are not orthogonal to each other and we must apply Gram-Schmidt. Applying Gram-Schmidt and normalizing gives us the following orthonormal basis of eigenvectors.

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus,

$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$