Math 54
Fall 2017
Practice Exam 2
Exam date: 10/31/17
Time Limit: 80 Minutes

Name: ______________________

Student ID: ______________________

GSI or Section: ______________________

This exam contains 7 pages (including this cover page) and 7 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

This is a closed book exam. No notes or calculators are permitted. We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space. Please do not tear out any pages.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

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1. (10 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the following vectors.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) (2 points) Check that the vectors v_1 and v_2 are orthogonal.

Solution: The dot product is

$$v_1 \cdot v_2 = 2 + 0 + 0 - 2 = 0.$$

(b) (4 points) Find the orthogonal projection of the vector $w = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ onto S.

Solution: We use the following formula.

$$\mathrm{proj}_S(w) = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2$$

The relevant inner products are as follows.

$$\begin{aligned} w \cdot v_1 &= 1 + 0 + 0 + 2 = 3 \\ w \cdot v_2 &= 2 - 2 + 0 - 1 = -1 \\ v_1 \cdot v_1 &= 1^2 + 0^2 + (-1)^2 + (-2)^2 = 6 \\ v_2 \cdot v_2 &= 2^2 + (-1)^2 + 0^2 + 1^2 = 6 \end{aligned}$$

Therefore we have

$$\mathrm{proj}_S(w) = \frac{3}{6} \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ -3 \\ -7 \end{bmatrix}.$$

(c) (4 points) Find a basis for the orthogonal complement of S.

Solution: The orthogonal complement of S is the null space of the matrix

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & 1 \end{bmatrix}.$$

Substracting twice the first row from the second one we get

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 5 \end{bmatrix}.$$

A basis for the nullspace is then

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) (3 points) Find the eigenvalues of A.

Solution:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 2 \\ 3 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = -\lambda(-\lambda)(1-\lambda)$$

So, the eigenvalues are $0, 1$.

(b) (4 points) Find bases of the eigenspaces of A.

Solution: We have that

$$A - 0I = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis of E_0.

We also have that

$$A - I = \begin{bmatrix} 0 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right\}$ is a basis of E_1

(c) (3 points) Is A diagonalizable?

Solution: By part (b), the sum of the dimensions of the eigenspaces of A is 2 which is less than 3 so A is not diagonalizable.

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point and a brief justification is worth 1 point. Credit for the justification can only be earned in conjunction with a correct answer. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

(a) (2 points) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^n$, and $[T]_\mathcal{B}$ and $[T]_\mathcal{C}$ are the $\mathcal{B}$ and $\mathcal{C}$ matrices of T, then $[T]_\mathcal{B} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[T]_\mathcal{C}$ where $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

Solution: False.
The correct formula is $[T]_\mathcal{B} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[T]_\mathcal{C}\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ which is clearly different.

(b) (2 points) If B is an echelon form of the matrix A, then the pivot columns of B form a basis of $\mathrm{Col}(A)$.

Solution: False.
For example, take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The pivot column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of B is not even in the column space of A.

(c) (2 points) If A is a 3×3 diagonalizable matrix whose only eigenvalues are 1 and 2, then $(A - I)(A - 2I) = 0$.

Solution: True.
We can write $A = PDP^{-1}$ where D is a diagonal matrix with either two 1's and a 2 on the diagonal or two 2's and a 1 on the diagonal. Since $(A - I)(A - 2I) = (PDP^{-1} - I)(PDP^{-1} - 2I) = P(D - I)(D - 2I)P^{-1}$, it is enough to check that $(D - I)(D - 2I) = 0$. We can write this last term as $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which are both 0.

(d) (2 points) The intersection of a subspace W of $\mathbb{R}^n$ and its orthogonal complement $W^\perp$ always has dimension 0.

Solution: True.
If v is in the intersection, then $v \cdot v = 0$ so $v = 0$. Hence, the intersection is always the trivial subspace $\{0\}$ which has dimension 0.

(e) (2 points) There is a linear transformation $T : \mathbb{P}_3 \to \mathbb{R}^2$ with $\dim \mathrm{Nul}(T) = 1$ where $\mathbb{P}_3$ is the vector space of polynomials of degree at most 3.

Solution: False.
The rank theorem would give that $\dim \mathbb{P}_3 = 4 = \dim \mathrm{Nul}(T) + \dim \mathrm{Im}(T) = 1 + \dim \mathrm{Im}(T)$ so $\dim \mathrm{Im}(T) = 3$. But, $\dim \mathrm{Im}(T) \leq \dim \mathbb{R}^2 = 2$ since $\mathrm{Im}(T) \subset \mathbb{R}^2$.

4. (a) (5 points) Suppose W is a subspace of $\mathbb{R}^n$. Show that the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$v \mapsto \text{Proj}_W(v)$$

is a linear transformation, where $\text{Proj}_W(v)$ is the orthogonal projection of v onto W.

Solution: Every vector $v \in \mathbb{R}^n$ can be written uniquely as

$$v = \hat{v} + v^\perp$$

where $\hat{v} = \text{Proj}_W(v) \in W$ and $v^\perp \in W^\perp$ by the orthogonal decomposition theorem. Let $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Then as $cv = c\hat{v} + cv^\perp$ and $W^\perp$ is also a subspace of $\mathbb{R}^n$, we must have $c\text{Proj}_W(v) = c\hat{v} = \text{Proj}_W(cv)$ by uniqueness of the orthogonal decomposition. Similarly, $v + w = \hat{v} + v^\perp + \hat{w} + w^\perp = \hat{v} + \hat{w} + v^\perp + w^\perp$. Again, as W and $W^\perp$ are both subspaces of $\mathbb{R}^n$, uniqueness of the orthogonal decomposition implies that $\text{Proj}_W(v + w) = \hat{v} + \hat{w} = \text{Proj}_W(v) + \text{Proj}_W(w)$. Thus, we have shown that T is a linear transformation.

Alternatively, choose an orthonormal basis $\{u_1, \ldots, u_r\}$ of W. Then $\text{Proj}_W(v) = \sum_{i=1}^r \frac{u_i \cdot v}{u_i \cdot u_i} u_i$. If $w \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have $u_i \cdot (cv) = c u_i \cdot v$ and $u_i \cdot (v + w) = u_i \cdot v + u_i \cdot w$ by the properties of the dot product so that

$$\text{Proj}_W(cv) = \sum_{i=1}^r \frac{u_i \cdot (cv)}{u_i \cdot u_i} u_i = \sum_{i=1}^r c \frac{u_i \cdot v}{u_i \cdot u_i} u_i = c \sum_{i=1}^r \frac{u_i \cdot v}{u_i \cdot u_i} u_i$$

and

$$\begin{aligned} \text{Proj}_W(v + w) &= \sum_{i=1}^r \frac{u_i \cdot (v + w)}{u_i \cdot u_i} u_i \\ &= \sum_{i=1}^r \frac{u_i \cdot v + u_i \cdot w}{u_i \cdot u_i} u_i \\ &= \sum_{i=1}^r \frac{u_i \cdot v}{u_i \cdot u_i} u_i + \sum_{i=1}^r \frac{u_i \cdot w}{u_i \cdot u_i} u_i. \end{aligned}$$

This implies that T is a linear transformation.

(b) (5 points) Show that there is always a basis $\mathcal{B}$ of $\mathbb{R}^n$ with respect to which T is diagonal.

Solution: We need $\mathcal{B}$ to be a basis of eigenvectors. Suppose that $\{w_1, \ldots, w_k\}$ is an orthogonal basis of W and $\{v_1, \ldots, v_{n-k}\}$ is an orthogonal basis of $W^\perp$. Then, $\{w_1, \ldots, w_k, v_1, \ldots, v_{n-k}\}$ is a basis of $\mathbb{R}^n$ since the vectors are all orthogonal. Further, these are all eigenvectors since

$$T(w_i) = \text{Proj}_W(w_i) = w_i \qquad \text{and} \qquad T(v_i) = \text{Proj}_W(v_i) = 0$$

since $w_i \in W$ and $v_i \in W^\perp$. Thus, $\mathcal{B} = \{w_1, \ldots, w_k, v_1, \ldots, v_{n-k}\}$ is one such basis.

5. (10 points) Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix A has the form $A = PCP^{-1}$.

$$A = \begin{bmatrix} -11 & -4 \\ 20 & 5 \end{bmatrix}$$

Solution: First, we need to find the eigenvalues of A. For that,

$$\det(A - \lambda I) = det(\begin{bmatrix} -11-\lambda & -4 \\ 20 & 5-\lambda \end{bmatrix}) = (-11-\lambda)(5-\lambda) + 20 \times 4 = \lambda^2 + 6\lambda + 25.$$

Thus, we have

$$\lambda^2 + 6\lambda + 25 = 0$$

And

$$\lambda = \frac{-6 \pm \sqrt{36 - 4 \times 25}}{2} = -3 \pm 4i.$$

Now, we have to calculate the corresponding eigenvector for $\lambda = -3 - 4i$. We have

$$A - \lambda I = \begin{bmatrix} -8+4i & -4 \\ 20 & 8+4i \end{bmatrix} \sim \begin{bmatrix} 20 & 8+4i \\ 0 & 0 \end{bmatrix}.$$

Thus, we get

$$20x_1 + (8+4i)x_2 = 0 \to x_1 = \frac{-2-i}{5}x_2$$

and an eigenvector is

$$v = \begin{bmatrix} -\frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}.$$

P can then be written as

$$P = \begin{bmatrix} Rev & Imv \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \\ 1 & 0 \end{bmatrix}$$

And finally, C can be written as

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}.$$

6. (10 points) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) (5 points) Find an orthogonal basis for $\mathrm{Col}\, A$.

Solution: The columns of A are visibly linearly independent, so we use the Gram-Schmidt process to turn the basis of $\mathrm{Col}\, A$ given by the columns of A into an orthogonal basis $\{v_1, v_2, v_3\}$ of $\mathrm{Col}\, A$. We find

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}v_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 1/2 \end{bmatrix};$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}v_1 - \frac{1}{3}v_2 = \begin{bmatrix} -1/2 + 1/6 \\ -1/3 \\ 0 \\ 1 - 1/2 - 1/6 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 0 \\ 1/3 \end{bmatrix}.$$

(b) (5 points) Find a 3×3 invertible upper triangular matrix R such that $A = QR$ where Q is a matrix whose columns form an orthogonal basis of Col(A).

Solution: Set $Q = \begin{bmatrix} 1 & -1/2 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \\ 1 & 1/2 & 1/3 \end{bmatrix}$. Q satisfies the desired condition by part (a), so since we want $A = QR$, we need $Q^T A = Q^T Q R$. Because the columns of Q are orthogonal, we have

$$Q^T Q = \begin{bmatrix} v_1 \cdot v_1 & 0 & 0 \\ 0 & v_2 \cdot v_2 & 0 \\ 0 & 0 & v_3 \cdot v_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Then

$$\begin{aligned} R &= (Q^T Q)^{-1} Q^T A \\ &= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1/2 & 1 & 0 & 1/2 \\ -1/3 & -1/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

7. (a) (5 points) Suppose that M is an $n \times n$ matrix such that $M^T = M$. If v_1 and v_2 are eigenvectors of M for eigenvalues λ_1 and λ_2, respectively, with $\lambda_1 \neq \lambda_2$, show that v_1 and v_2 are orthogonal.

Solution: We have

$$\lambda_1(v_1 \cdot v_2) = Mv_1 \cdot v_2 = v_1 \cdot M^T v_2 = v_1 \cdot Mv_2 = \lambda_2(v_1 \cdot v_2).$$

Thus,

$$(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0,$$

and we must have $v_1 \cdot v_2 = 0$ since $\lambda_1 - \lambda_2 \neq 0$.

(b) (5 points) Suppose that A and B are $n \times n$ matrices. Suppose $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of $\mathbb{R}^n$ such that each b_i is an eigenvector of both A and B. Show that $AB = BA$.

Solution: Let P be the matrix whose columns are $b_1, \dots, b_n$. Then $A = PD_1P^{-1}$ and $B = PD_2P^{-1}$ where D_1 and D_2 are diagonal matrices. Then,

$$AB = PD_1P^{-1}PD_2P^{-1} = PD_1D_2P^{-1} = PD_2D_1P^{-1} = PD_2P^{-1}PD_1P^{-1} = BA$$

as diagonal matrices always commute with each other since their multiplication is simply given by multiplying the entries on the diagonal.

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