Math 54
Fall 2017
Practice Exam 1
Exam date: 9/26/17
Time Limit: 80 Minutes

Name: ____________________

Student ID: ____________________

GSI or Section: ____________________

This exam contains 6 pages (including this cover page) and 7 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

> This is a closed book exam. No notes or calculators are permitted.
> We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

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1. (a) (5 points) Determine if $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 \\ -1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ 2 \\ -3 \\ 2 \end{bmatrix}$.

Solution: We may check whether $\mathbf{b}$ is a linear combination of the given vectors by checking if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, where A is the matrix whose columns are the given vectors. To do this, we row reduce the augmented matrix $[A \mid \mathbf{b}]$:

$$\begin{bmatrix} 3 & 2 & 0 & -3 & 1 \\ -1 & 4 & 0 & 2 & 0 \\ 2 & 8 & 1 & -3 & 0 \\ 3 & 16 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 4 & 0 & 2 & 0 \\ 3 & 2 & 0 & -3 & 1 \\ 2 & 8 & 1 & -3 & 0 \\ 3 & 16 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 4 & 0 & 2 & 0 \\ 0 & 14 & 0 & 3 & 1 \\ 0 & 16 & 1 & 1 & 0 \\ 0 & 28 & 0 & 8 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} -1 & 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3/14 & 1/14 \\ 0 & 16 & 1 & 1 & 0 \\ 0 & 28 & 0 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3/14 & 1/14 \\ 0 & 0 & 1 & -17/7 & -8/7 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

We see that there is no pivot in the last column, so the equation $A\mathbf{x} = \mathbf{b}$ is consistent, and $\mathbf{b}$ is indeed a linear combination of the given vectors.

(b) (5 points) Without doing any further computations, determine if $A = \begin{bmatrix} 3 & 2 & 0 & -3 \\ -1 & 4 & 0 & 2 \\ 2 & 8 & 1 & -3 \\ 3 & 16 & 0 & 2 \end{bmatrix}$ is invertible. Then, compute $\det(A)$ to verify your conclusion.

Solution: From the computations in the first part, we see that A has a pivot in every row and column so A is invertible. Now we compute the deteriminant. We get

$$\begin{vmatrix} 3 & 2 & 0 & -3 \\ -1 & 4 & 0 & 2 \\ 2 & 8 & 1 & -3 \\ 3 & 16 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 & -3 \\ -1 & 4 & 2 \\ 3 & 16 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 14 & 3 \\ -1 & 4 & 2 \\ 0 & 28 & 8 \end{vmatrix} = \begin{vmatrix} 14 & 3 \\ 28 & 8 \end{vmatrix} = 112 - 84 = 28$$

using a combination of cofactor expansion and row operations. Since $\det(A) \neq 0$ this confirms that A is invertible.

2. (a) (5 points) Suppose that the augmented matrix corresponding to a linear system of equations can be row reduced to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 5 \\ 0 & 1 & 7 & -2 \end{array}\right]$$

Describe the solution set in parametric vector form. Then state what shape it represents geometrically; for example, your answer could be of the form: a plane, a sphere, etc.

Solution: x_3 is a free variable and x_1 and x_2 can be written in terms of x_3. Setting $x_3 = t$, we have that the parametric vector form is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix} t$$

for $t \in \mathbb{R}$. This is a <u>line</u> through the point $(5, -2, 0)$ parallel to the vector $\begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}$.

(b) (5 points) Describe the span of $v_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix}$ geometrically in the same way as in part (a).

How many vectors (if any at all) can be removed from the collection $\{v_1, v_2, v_3\}$ so that the resulting collection of vectors has the same span as $\{v_1, v_2, v_3\}$?

Solution: To describe the span geometrically and determine how many vectors are needed, we need to see if these vectors are linearly independent. To determine if the set is linearly independent, we need to see if the matrix with these vectors as columns has a pivot in every column. We row reduce

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 4 \\ 3 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

to see that there are two pivots. That means there is one linear relation between v_1, v_2, and v_3. This means that their span is a plane in $\mathbb{R}^3$, and one vector can be removed from the original collection without changing the span.

3. (10 points) Consider the matrices below.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

(a) (3 points) Compute $\det(AB)$.

Solution: We have

$$\det(AB) = \det(A)\det(B) = 3 \cdot 25 = 75.$$

(b) (3 points) Compute the determinant of the matrix C obtained from A by switching the first and third row, then adding twice the second column to the first, and then multiplying the second and third rows by 2.

Solution: The first operation changes the sign of the determinant. The second does not change the determinant. The third is two row operations which each multiply the determinant by 2. Thus,

$$\det(C) = -2^2 \det(A) = -12.$$

(c) (4 points) Do A and B commute?

Solution: By direct computation, we get

$$AB = \begin{bmatrix} 5 & 4 & 3 \\ 10 & 11 & 12 \\ 10 & 9 & 13 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 19 & 15 & 3 \\ 6 & 5 & 2 \\ 10 & 5 & 5 \end{bmatrix}.$$

Since $AB \neq BA$, the matrices do not commute.

4. (10 points) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix}.$$

(a) (3 points) Compute the matrix of T.

Solution: The matrix is

$$\left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b) (2 points) Write the definition of what it means for T to be onto.

Solution: T is onto if for every $\mathbf{b} \in \mathbb{R}^3$, there exists $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$.

(c) (1 point) Is T onto?

Solution: No. T cannot be onto since its matrix has more rows than columns and thus cannot have a pivot in every row.

(d) (2 points) Write the definition of what it means for T to be one-to-one.

Solution: T is one-to-one if whenever $T(\mathbf{x}) = T(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then $\mathbf{x} = \mathbf{y}$.

(e) (2 points) Is T one-to-one?

Solution: We need to check if the matrix of T has a pivot in every column. Row reducing gives

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

so there is indeed a pivot in every column. Therefore, T is one-to-one.

5. (10 points) Label the following statements as True or False. The correct answer is worth 1 point. An additional point will be awarded for a correct brief justification. No points will be rewarded if it is not clear whether you intended to mark the statement as True or False.

(a) (2 points) A homogeneous system is always consistent.

Solution: True.
$\mathbf{x} = \mathbf{0}$ is always a solution to a homogeneous system.

(b) (2 points) Let RREF(C) be the reduced row echelon form of a matrix C. If A and B are matrices such that AB is well defined, then RREF(AB)= RREF(A)RREF(B).

Solution: False.
Consider the following matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Then $\text{RREF}(A) = \text{RREF}(B) = A$. But, $A^2 = A$ and $AB = 0$.

(c) (2 points) Assume A and B are invertible. If $AB = BA$, then $A^{-2}B^{-2} = B^{-2}A^{-2}$ where $A^{-2} = (A^{-1})^2$.

Solution: True.
Since $AB = BA$, taking $(\cdot)^{-1}$ on both side gives $B^{-1}A^{-1} = A^{-1}B^{-1}$. Therefore,

$$\begin{aligned} A^{-2}B^{-2} &= A^{-1}(A^{-1}B^{-1})B^{-1} = A^{-1}(B^{-1}A^{-1})B^{-1} = (B^{-1}A^{-1})A^{-1}B^{-1} \\ &= B^{-1}A^{-1}(B^{-1}A^{-1}) = B^{-1}(B^{-1}A^{-1})A^{-1} = B^{-2}A^{-2}. \end{aligned}$$

(d) (2 points) If A is an invertible $n \times n$ matrix then $A^{-1} = \frac{1}{\det(A)}C$, where C is the matrix of A's cofactors.

Solution: False.
We have that $A^{-1} = \frac{1}{\det(A)}C^T$ and there are many examples where $C \neq C^T$ such as

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(e) (2 points) If $k < n$, then a set of k vectors in $\mathbb{R}^n$ is necessarily linearly independent.

Solution: False.
Take any of the vectors in the set to be the zero vector and the set will be linearly dependent.

6. (10 points) Using $\blacksquare$ to denote a leading entry, $*$ to denote an arbitrary nonleading entry, and 0 to denote an entry that must be 0, describe all 2×3 row echelon form (REF) matrices. (*Hint*: There are seven.)

   For each matrix, say whether it has linearly independent or linearly dependent columns and whether or not its columns span $\mathbb{R}^2$.

Solution: The seven possibilites are

$$(1) \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix} \qquad (2) \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{bmatrix} \qquad (3) \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$$

$$(4) \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & 0 \end{bmatrix} \qquad (5) \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix} \qquad (6) \begin{bmatrix} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

$$(7) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

All have linearly dependent columns since they have more columns than rows. (1), (2), and (3) have columns that span $\mathbb{R}^2$ since there is a pivot in each row. All of the others have columns that do not span $\mathbb{R}^2$.

7. (10 points) Suppose that A is an invertible $n \times n$ matrix with integer entries.

   (a) (5 points) Show that if $\det(A) = \pm 1$, then A^{-1} has integer entries.

   Solution: We have that
   $$A^{-1} = \frac{1}{\det(A)} C^T$$
   where C is the matrix of cofactors for A. Since the entries of C^T are determinants of matrices with integer entries, C^T will have integer entries. Since $\det(A) = \pm 1$, $1/\det(A)$ is an integer. Therefore, A^{-1} has integer entries.

   (b) (5 points) Show that if A^{-1} has integer entries, then $\det(A) = \pm 1$. (*Hint*: Consider $\det(A^{-1})$.)

   Solution: We have that $\det(A^{-1}) = \frac{1}{\det(A)}$. But, $\det(A^{-1})$ must be an integer since A^{-1} has integer entries. The only way that $\frac{1}{\det(A)}$ is an integer is if $\det(A) = \pm 1$ as we wanted to show.

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