Math 54
Fall 2017
Practice Final Exam
Exam date: 12/14/17
Time Limit: 170 Minutes

Name: _____

Student ID: _____

GSI or Section: _____

This exam contains 9 pages (including this cover page) and 10 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

> This is a closed book exam. No notes or calculators are permitted.
> We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space. Please do not tear out any pages.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

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1. (10 points) Consider the matrix

$$A_x = \begin{bmatrix} 1 & 1 & 2 \\ x & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) (5 points) Find all values of x such A_x is invertible.

Solution: We compute the determinant.

$$\begin{vmatrix} 1 & 1 & 2 \\ x & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 \\ x & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ x & 2 \end{vmatrix} = x - 1$$

Therefore, A_x is invertible as long as $x \neq 1$.

(b) (5 points) Compute A_2^{-1}.

Solution: We proceed by row reduction.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

Therefore,

$$A_2^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

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2. (10 points) Consider the matrices

$$A = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

(a) (2 points) Which of these matrices is orthogonally diagonalizable?

Solution: A is orthogonally diagonalizable since A is symmetric. B is not orthogonally diagonalizable since B is not symmetric.

(b) (2 points) Find the eigenvalues of the orthogonally diagonalizable matrix from part (a)

Solution: We have

$$\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & -1 & 2 \\ -1 & 5 - \lambda & 2 \\ 2 & 2 & 2 - \lambda \end{vmatrix}$$

$$= (5 - \lambda)\begin{vmatrix} 5 - \lambda & 2 \\ 2 & 2 - \lambda \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 2 & 2 - \lambda \end{vmatrix} + 2\begin{vmatrix} -1 & 5 - \lambda \\ 2 & 2 \end{vmatrix}$$

$$= (5 - \lambda)(\lambda^2 - 7\lambda + 6) + \lambda - 6 + -24 + 4\lambda$$

$$= -\lambda^3 + 12\lambda^2 - 36\lambda$$

$$-\lambda(\lambda^2 - 12\lambda + 36) = -\lambda(\lambda - 6)^2$$

so the eigenvalues are 0 and 6.

(c) (6 points) Find an orthogonal matrix P consisting of eigenvectors of the orthogonally diagonalizable matrix from part (a).

Solution: We start by finding an orthogonal basis of each eigenspace. For E_0, we find the null space of A.

$$\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -2 \\ 0 & 24 & 12 \\ 0 & 12 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis for E_0. For E_6, we find the null space of $A - 6I$

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for E_6. Applying the Gram-Schmidt algorithm gives

that $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for E_6. Normalizing, we get

$$P = \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}$$

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point and a brief justification is worth 1 point. Credit for the justification can only be earned in conjunction with a correct answer. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

(a) (2 points) If σ is the largest singular value of an $m \times n$ matrix A, then $\|Av\| \leq \sigma\|v\|$ for all $v \in \mathbb{R}^n$ where $\|u\| = \sqrt{u \cdot u}$.

> **Solution:** True.
> By the singular value decomposition, there is an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ and an orthonormal basis $\{u_1, \ldots, u_m\}$ of $\mathbb{R}^m$ such that $Av_i = \sigma_i u_i$ and $\sigma_i \leq \sigma$ are the singular values of A. Then, For any $v = c_1 v_1 + \ldots + c_n v_n$, we have
>
> $$Av \cdot Av = c_1^2 \sigma_1^2 + \ldots + c_n^2 \sigma_n^2 \leq \sigma^2(c_1^2 + \ldots + c_n^2) = \sigma^2 v \cdot v$$
>
> as desired.

(b) (2 points) If A and B are similar $n \times n$ matrices so that $A = PBP^{-1}$, then $Py(t)$ is a solution to $x'(t) = Ax(t)$ for any $y(t)$ such that $y'(t) = By(t)$.

> **Solution:** True.
> We have $y'(t) = By(t) = P^{-1}APy(t)$. After applying P to both sides, we get $Py'(t) = APy(t)$ as we needed since $\frac{d}{dt}Py(t) = Py'(t)$.

(c) (2 points) Only square matrices can be squared.

> **Solution:** True.
> In order for A^2 to make sense for an $m \times n$ matrix, we must have $m = n$.

(d) (2 points) If $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$ such that $\langle v, v \rangle = v \cdot v$ for all $v \in \mathbb{R}^n$ then $\langle u, w \rangle = u \cdot w$ for all $u, w \in \mathbb{R}^n$.

> **Solution:** True.
> By properties of inner products, we have $\langle u, w \rangle = \frac{1}{2}\left(\langle u + w, u + w \rangle - \langle u, u \rangle - \langle w, w \rangle\right) = \frac{1}{2}((u + w) \cdot (u + w) - u \cdot u - w \cdot w) = u \cdot w$.

(e) (2 points) The set of solutions to $y''' + by'' + cy' + dy = 0$ is always a vector space of dimension three.

> **Solution:** True.
> The solutions of a homogeneous equation always form a vector space so we need to check the dimension. The solutions of this equation can recast as solutions to an equation of the form $x'(t) = Ax(t)$ where A is 3×3 matrix by setting $x_1 = y, x_2 = y'$, and $x_3 = y''$. Such an equation always has a 3-dimensional solution space spanned by the columns of a fundamental matrix (such as e^{At}).

4. (10 points) Consider the function on $\mathbb{R}^3$ given by

$$x \star y = \sum_{m=1}^{3} m x_m y_m$$

for $x, y \in \mathbb{R}^3$.

(a) (3 points) Show that this is an inner product on $\mathbb{R}^3$.

Solution: We need to check four properties: $x \star y = y \star x$, $(x + v) \star y = x \star y + v \star y$, $(cx) \star y = c(x \star y)$, and $x \star x \geq 0$ and only zero when $x = 0$ for $x, y, v \in \mathbb{R}^3$ and $c \in \mathbb{R}$. For the first, we have

$$x \star y = \sum_{m=1}^{3} m x_m y_m = \sum_{m=1}^{3} m y_m x_m = y \star x.$$

For the second,

$$(x + v) \star y = \sum_{m=1}^{3} m(x_m + v_m) y_m = \sum_{m=1}^{3} m x_m y_m + \sum_{m=1}^{3} m v_m y_m = x \star y + v \star y.$$

For the third,

$$(cx) \star y = \sum_{m=1}^{3} m c x_m y_m = c \left(\sum_{m=1}^{3} m x_m y_m \right) = c(x \star y).$$

For the last

$$x \star x = \sum_{i=1}^{3} m x_m^2 \geq 0$$

since each term is positive and equal to zero only when $x_1^2 = x_2^2 = x_3^2 = 0$, i.e., $x = 0$. We have verified that this is an inner product.

(b) (4 points) Find an orthogonal basis for $V = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ with respect to the $\star$ inner product.

Solution: We apply the Gram-Schmidt procedure. We have

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \star \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \star \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{-1}{6} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ 7 \\ -5 \end{bmatrix}$$

form an orthogonal basis of V

(c) (3 points) Find the closest vector in V (from (b)) to $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ when the distance is computed with the $\star$ inner product.

Solution: The closest vector is the orthogonal projection. To simplify computations, we replace u_1, u_2 from part (b) with the orthogonal basis $w_1 = u_1, w_2 = 6u_2$. We have

$$\mathrm{Proj}_V(x) = \frac{x \star w_1}{w_1 \star w_1} w_1 + \frac{x \star w_2}{w_2 \star w_2} w_2 = \frac{1}{2} w_1 - \frac{5}{58} w_2 = \frac{1}{58} \begin{bmatrix} 24 \\ -6 \\ 54 \end{bmatrix}.$$

5. (10 points) Suppose that U is an orthogonal $n \times n$ matrix that is orthogonally diagonalizable.

   (a) (4 points) Show that the only eigenvalues of U are ± 1.

   > **Solution:** Since U is orthogonal, $U^T = U^{-1}$. Since U is orthogonal diagonalizable, U is symmetric $U^T = U$. Thus, we have $U^2 = I$. If v is an eigenvector with eigenvalue λ, we have
   > $$\lambda^2 v = U^2 v = v$$
   > so $\lambda^2 = 1$. That is, $\lambda = \pm 1$ as desired.

   (b) (3 points) Show that if U has only 1 as an eigenvalue then U is the $n \times n$ identity matrix.

   > **Solution:** In that case, we can diagonalize U to get
   > $$U = PIP^{-1} = PP^{-1} = I$$
   > since the only eigenvalue of U is 1.

   (c) (3 points) Give an example of an orthogonal and orthogonally diagonalizable matrix U with entries other than ± 1.

   > **Solution:** One simple example is
   > $$U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

6. (10 points) Suppose that y_1 and y_2 are solutions to $y'' + b(t)y' + c(t)y = 0$ and $W(t) = W[y_1, y_2](t)$ is their Wronskian.

   (a) (4 points) Using only the defintion of $W(t)$, show that $W(t)$ satisfies the equation $W'(t) = -b(t)W(t)$. Deduce from your calculation that if $W(0) \neq 0$ then $W(t) \neq 0$ for all t.

   Solution: We have that

   $$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

   Differentiating, we get

   $$W'(t) = y_1'(t)y_2'(t) + y_1(t)y_2''(t) - y_2'(t)y_1'(t) - y_2(t)y_1''(t)$$

   $$= y_1(t)\big[- b(t)y_2'(t) - c(t)y_2(t)\big] - y_2(t)\big[- b(t)y_1'(t) - c(t)y_1(t)\big]$$

   $$= b(t)(y_1'(t)y_2(t) - y_2'(t)y_1(t) = -b(t)W(t)$$

   as desired. This computation shows that

   $$W(t) = Ce^{-\int_0^t b(s)\, ds}$$

   is never zero if $W(0) = C \neq 0$.

   (b) (2 points) Show that if $y_1(t) \neq 0$ then

   $$\left(\frac{y_2}{y_1}\right)' = \frac{W}{(y_1)^2}$$

   Solution: We compute

   $$\left(\frac{y_2}{y_1}\right)' = \frac{y_2'y_1 - y_1'y_2}{(y_1)^2} = \frac{W}{(y_1)^2}$$

   using the product rule.

(c) (4 points) Given that $y_1(t) = e^{-t^2}$ is a solution to $y'' + 4ty' + (4t^2 + 2)y = 0$, find another linearly independent solution using parts (a) and (b).

Solution: By part (a),

$$W(t) = Ce^{-\int_0^t b(s)\,ds} = Ce^{-2t^2}.$$

By rescaling the unknown linearly independent solution, we can assume that $C = 1$. By part (b), we get that

$$\left(\frac{y_2}{y_1}\right)' = \frac{e^{-2t^2}}{e^{-2t^2}} = 1.$$

Thus, we can take

$$y_2(t) = ty_1(t) = te^{-t^2}.$$

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7. (10 points) Consider the differential equation $y'''' - 6y''' + 14y'' - 14y' + 5y = 0$.

   (a) (6 points) Find the general solution of the above differential equation. (*Hint:* $r = 2 + i$ is a root of the characteristic polynomial.)

   > **Solution:** By the standard substitution $x_1 = y, x_2 = y', x_3 = y''$, and $x_4 = y'''$, we can translate this to an equation of the form $x'(t) = Ax(t)$ where A is 4×4 matrix with characteristic polynomial is $r^4 - r^3 + 14r^2 - 14r + 5 = 0$. We know that $2 + i$ is a root so $2 - i$ must also be a root since we are working with a real polynomial. Therefore, $(r - (2 + i))(r - (2 - i)) = r^2 - 4r + 5$ divides the polynomial. Doing long division gives that
   >
   > $$r^4 - r^3 + 14r^2 - 14r + 5 = (r^2 - 4r + 5)(r^2 - 2r + 1)$$
   >
   > so the only other root is the double root $r = 1$. Thus, the general solution is
   >
   > $$y(t) = C_1 e^{2t} \cos(t) + C_2 e^{2t} \sin(t) + C_3 e^t + C_4 t e^t.$$

   (b) (4 points) Solve the initial value problem where $y(0) = 1, y'(0) = 3, y''(0) = 5$, and $y'''(0) = 5$.

   > **Solution:** From the general solution in the first part, we calculate
   >
   > $$y'(t) = (2C_1 + C_2)e^{2t} \cos(t) + (-C_1 + 2C_2)e^{2t} \sin(t) + (C_3 + C_4)e^t + C_4 t e^t,$$
   > $$y''(t) = (3C_1 + 4C_2)e^{2t} \cos(t) + (-4C_1 + 3C_2)e^{2t} \sin(t) + (C_3 + 2C_4)e^t + C_4 t e^t,$$
   > $$y'''(t) = (2C_1 + 11C_2)e^{2t} \cos(t) + (-11C_1 + 2C_2)e^{2t} \sin(t) + (C_3 + 3C_4)e^t + C_4 t e^t.$$
   >
   > By putting $t = 0$, we get a system of linear equations corresponding to an augmented matrix which we row reduce below.
   >
   > $$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 1 \\ 2 & 1 & 1 & 1 & | & 3 \\ 3 & 4 & 1 & 2 & | & 5 \\ 2 & 11 & 1 & 3 & | & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & 1 \\ 0 & 1 & -1 & 1 & | & 1 \\ 0 & 4 & -2 & 2 & | & 2 \\ 0 & 11 & -2 & 3 & | & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & 1 \\ 0 & 1 & -1 & 1 & | & 1 \\ 0 & 0 & 2 & -2 & | & -2 \\ 0 & 0 & 10 & -8 & | & -8 \end{bmatrix}$$
   >
   > $$\longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & 1 \\ 0 & 1 & -1 & 1 & | & 1 \\ 0 & 0 & 2 & -2 & | & 2 \\ 0 & 0 & 0 & 2 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 1 \end{bmatrix}.$$
   >
   > So the solution to the IVP is $y(t) = e^{2t} \cos(t) + t e^t$.

8. (a) (4 points) Find a basis of solutions to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t)$$

Solution: Since $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ is upper triangular, its eigenvalues are the entries on the diagonal. We compute the eigenspaces.

$$E_1 = \text{Nul}(A - \text{Id}) = \text{Nul} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_2 = \text{Nul}(A - 2\text{Id}) = \text{Nul} \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore a basis of solutions to the given system is

$$\left\{ e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

(b) (6 points) Find a particular solution to

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} t^{-1}e^{2t} \\ 0 \end{bmatrix}$$

Solution: Variation of parameters tells us that we can find a solution of the form $X(t)v(t)$ where

$$X(t) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 0 \end{bmatrix}$$

and $v(t)$ is a vector satisfying

$$X(t)v'(t) = \begin{bmatrix} t^{-1}e^{2t} \\ 0 \end{bmatrix}$$

We therefore find that $v'(t) = \begin{bmatrix} 0 \\ t^{-1} \end{bmatrix}$, and so we can take $v(t) = \begin{bmatrix} 0 \\ \log|t| \end{bmatrix}$. This gives us the solution

$$X(t)v(t) = \begin{bmatrix} e^{2t}\log|t| \\ 0 \end{bmatrix}$$

9. (10 points) Consider the heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}.$$

(a) (6 points) Find the solution defined for $0 \le x \le \pi$ and $t \ge 0$ that satisfies $u(0,t) = u(\pi, t) = 0$ and $u(x, 0) = \sin(3x) - 5\sin(5x)$.

> **Solution:** The general solution with $L = \pi$ and $\beta = 2$ and Dirichlet boundary condition is given by
>
> $$\sum_{n=1}^{\infty} c_n \sin(nx)e^{-2n^2 t}.$$
>
> For $u(x, 0) = \sin(3x) - 5\sin(5x)$, we must have that the c_n are the coefficients of the Fourier sine series of this function. However, $\sin(3x) - 5\sin(5x)$ is already a sine series so we get $c_3 = 1, c_5 = -5$ and all the other coefficients vanish. Therefore,
>
> $$u(x, t) = \sin(3x)e^{-18t} - 5\sin(5x)e^{-50t}.$$

(b) (4 points) Suppose that $v(x, t)$ is the solution defined for $0 \le x \le \pi$ and $t \ge 0$ that satisfies $v(0, t) = 0$, $v(\pi, t) = 1$ and $v(x, 0) = \sin(3x) - 5\sin(5x)$. Find $\lim\limits_{t \to \infty} v(x, t)$.

> **Solution:** We can write
> $$v(x, t) = w(x, t) + \frac{x}{\pi}$$
>
> where $w(x, t)$ is a solution to the heat equation satisfying the Dirichlet boundary conditions and $w(x, 0) = \sin(3x) - 5\sin(5x) - \frac{x}{\pi}$. Thus, we see that $v(x, t)$ will be of the form
> $$v(x, t) = \frac{x}{\pi} + \sum_{n=1}^{\infty} b_n \sin(nx)e^{-2n^2 t}$$
>
> where the b_n are the coefficients of the Fourier sine series for $w(x, 0)$. Therefore,
>
> $$\lim_{t \to \infty} v(x, t) = \frac{x}{\pi} + \lim_{t \to \infty} \sum_{n=1}^{\infty} b_n \sin(nx)e^{-2n^2 t} = \frac{x}{\pi} + 0 = \frac{x}{\pi}.$$

10. (10 points) Consider the function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Euler famously computed $\zeta(2) = \pi^2/6$ in 1738. Evaluate $\zeta(4)$ by computing the Fourier series of $f : [-\pi, \pi] \to \mathbb{R}$, given by $f(x) = \pi^4 - x^4$. You may assume Euler's result.

Solution: Since f is even, $b_n = 0$ for all n. We now compute the other coefficients. We have

$$a_0 = \int_{-\pi}^{\pi} (\pi^4 - x^4)\, dx = \pi^4 x - \frac{x^5}{5} \bigg|_{-\pi}^{\pi} = 2\pi^5 - \frac{2\pi^5}{5} = \frac{8\pi^5}{5}$$

and for $n \geq 1$,

$$\int_{-\pi}^{\pi} (\pi^4 - x^4) \cos nx\, dx = \pi^4 \int_{-\pi}^{\pi} \cos nx\, dx - \int_{-\pi}^{\pi} x^4 \cos nx\, dx$$

$$= -\left[\frac{x^4 \sin nx}{n} \bigg|_{-\pi}^{\pi} - \frac{4}{n} \int_{-\pi}^{\pi} x^3 \sin nx\, dx \right]$$

$$= \frac{4}{n} \left[-\frac{x^3 \cos nx}{n} \bigg|_{-\pi}^{\pi} + \frac{3}{n} \int_{-\pi}^{\pi} x^2 \cos nx\, dx \right]$$

$$= \frac{4}{n} \left[-\left(\frac{\pi^3 \cos n\pi - (-\pi)^3 \cos(-n\pi)}{n} \right) + \frac{3}{n} \int_{-\pi}^{\pi} x^2 \cos nx\, dx \right]$$

$$= \frac{4}{n} \left[-\left(\frac{(-1)^n 2\pi^3}{n} \right) + \frac{3}{n} \int_{-\pi}^{\pi} x^2 \cos nx\, dx \right]$$

$$= \frac{(-1)^{n+1} 8\pi^3}{n^2} + \frac{12}{n^2} \left[\frac{x^2 \sin nx}{n} \bigg|_{-\pi}^{\pi} - \frac{2}{n} \int_{-\pi}^{\pi} x \sin nx\, dx \right]$$

$$= \frac{(-1)^{n+1} 8\pi^3}{n^2} - \frac{24}{n^3} \left[\frac{-x \cos nx}{n} \bigg|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos nx\, dx \right]$$

$$= \frac{(-1)^{n+1} 8\pi^3}{n^2} + \frac{24}{n^3} \left[\left(\frac{\pi \cos n\pi - (-\pi) \cos(-n\pi)}{n} \right) \right]$$

$$= (-1)^{n+1} \pi \frac{8(n^2\pi^2 - 6)}{n^4}.$$

Thus,

$$\frac{4\pi^4}{5} - 8 \sum_{n=1}^{\infty} (-1)^n \frac{n^2\pi^2 - 6}{n^4} \cos nx$$

is the Fourier series of f, and because the 2π-periodic extension of f is piecewise differentiable and continuous at π with $f(\pi) = 0$, we have

$$0 = f(\pi) = \frac{4\pi^4}{5} - 8\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} + 48 \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Rearranging the above, it follows that

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{48} \left(8\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{4\pi^4}{5} \right) = \frac{1}{48} \left(8\pi^2 \cdot \zeta(2) - \frac{4\pi^4}{5} \right) = \frac{\pi^4}{90}.$$

Extra space.

Extra space.

Extra space.