1. Consider the linear transformation which rotates the plane by $\pi/2$ degrees clockwise:

$$R : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\mathbf{x} \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

(a) Find the eigenvalues of $[R]$

> **Solution:** The characteristic polynomial is $\lambda^2 + 1$ which has roots $\lambda = \pm i$

(b) For each eigenvalue find a basis for its eigenspace

> **Solution:** For $\lambda = -i$ We need a basis for $\mathrm{Nul}([R] - (-i)I) = \mathrm{Nul}(\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$. This has basis $\{\begin{bmatrix} i \\ 1 \end{bmatrix}\}$. Taking conjugates we find a basis for the eigenspace of eigenvalue $\lambda = i$ is $\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\}$.

2. For each, give an example of the following, or explain why it can't exist:

(a) A 3×3 matrix, A with real entries but no real eigenvalues.

> **Solution:** This can't happen, as the characteristic polynomial of A has degree 3, and any odd degree polynomial with real coefficients has to have a real root.

(b) A 3×3 matrix with real entries and exactly 1 real eigenvalue.

> **Solution:** This can happen, for example if the characteristic polynomial was $-\lambda(\lambda^2 + 1)$, which has exactly one real root. This happens for example if $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.