Name: ______________________________

Section: ______________________________

1. Consider the basis $\mathcal{B} = \{1 - x^2, 2 + 4x + x^2, -4x - 2x^2\}$ of the space of polynomials with degree less than or equal to 2 with real coefficeints, $\mathcal{P}_2$. Find the coordinates of $x^2 + x + 1$ in this basis.

Solution: We are trying to find $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1(1 - x^2) + a_2(2 + 4x + x^2) + a_3(-4x - 2x^2) = 1 + x + x^2.$$

If we use the standard basis $\{1, x, x^2\}$ to transform this into a problem in $\mathbb{R}^3$, we are trying to solve

$$a_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, we proceed by row reduction (or your preferred method). We have

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & -4 & 1 \\ -1 & 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1/4 \\ 0 & 3 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 1 & -1 & 1/4 \\ 0 & 0 & 1 & 5/4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \end{pmatrix}.$$

Therefore, coordinates of $x^2 + x + 1$ with respect to $\mathcal{B}$ are

$$\begin{pmatrix} -2 \\ 3/2 \\ 5/4 \end{pmatrix}.$$

2. Consider the linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(A) = BA$ where B is the matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Calculate the matrix of T with respect to the standard basis for $M_{2\times 2}$, i.e., the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Solution: We compute

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix},$$

$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix},$$

$$T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the matrix for T with respect to the standard basis is

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$