1. (10 points) Write the definition of each of the following concepts. Use complete sentences and be as precise as you can.
   (a) (2 points) The inverse of a matrix.
   (b) (2 points) The set of vectors $\{v_1, \dots v_k\}$ in a vector space V being *linearly independent*.
   (c) (2 points) The dimension of a (finite-dimensional) vector space. (State the theorems which make this definition meaningful.)
   (d) (2 points) The projection of a vector v in $\mathbb{R}^n$ onto a subspace W.
   (e) (2 points) A diagonalizable matrix.

2. (10 points) Find the equation $y = \alpha + \beta x$ of the least-squares line that best fits the data

$$(a_1, b_1) = (0, 1), (a_2, b_2) = (1, 1), (a_3, b_3) = (1, 2)$$

That is, the equation minimizing $\sum_{n=1}^{n=3} |b_i - (\alpha + \beta a_i)|^2$.

3. (10 points) A matrix is called *nilpotent* if $A^n = 0$ for some $n > 0$.
   (a) (2 points) Write down an example of a nonzero nilpotent matrix.
   (b) (4 points) Show that the only eigenvalue of a nilpotent matrix is zero.
   (c) (2 points) If A is an $n \times n$ nilpotent matrix, what is the characteristic polynomial of A?
   (d) (2 points) Is $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ a nilpotent matrix?

4. (10 points) Consider the following five functions.

   1. $\det : M_{2\times 2} \to \mathbb{R}$ given by taking a matrix A to its determinant $\det(A)$.
   2. $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$.
   3. $e : M_{2\times 2} \to M_{2\times 2}$ given by $e(A) = e^A$.
   4. $S : \mathbb{R}^7 \to \mathbb{R}$ given by $S(v) = v \cdot w$ where $w \in \mathbb{R}^7$ is a fixed nonzero vector.
   5. $\Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $\Delta(y(x)) = y''(x)$.

   Here, $M_{2\times 2}$ is the space of 2×2 real matrices, $C^\infty(\mathbb{R})$ is the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and e^A is the matrix exponential.

   (a) (3 points) Which of these maps are linear transformations?

   (b) (3 points) Of the maps that are linear, which are injective?

   (c) (3 points) Of the maps that are linear, which are surjective?

   (d) (1 point) Of the maps that are linear, which are isomorphisms?

5. (10 points) Solve the following second order linear differential equation:

$$y'' - 3y' + 2y = 10t\sin(t) - 4\cos(t)$$

subject to the initial conditions:

$$y(0) = 12, \quad y'(0) = 15$$

6. (10 points) Give the general solution to the following differential equation system:

$$\mathbf{y}'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{y}(t)$$

We present three solutions because we are very kind.

7. (10 points)

(a) (6 points) Compute the Fourier series for the extension of

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

as a 2π periodic function.

(b) (4 points) Use part (a) to find the sum of the convergent series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}.$$

8. This problem concerns solutions to the heat equation

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with *periodic* boundary conditions $u(-L,t) = u(L,t)$, $\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$.

(a) (6 points) Using separation of variables $u(x,t) = X(x)T(t)$ as usual, we end up having to solve

$$X''(x) + \lambda X(x) = 0 \tag{2}$$

with periodic boundary conditions $X(-L) = X(L), X'(-L) = X'(L)$. For which $\lambda \geq 0$ does this boundary value problem have a nonzero solution, and what are those nonzero solutions?

(b) (4 points) What are the corresponding nonzero solutions to the heat equation?

9. (10 points) The Laplace equation is an important class of Partial Differential Equations that are very prevalent in many physical problems (E&M, Fluid, Mechanics etc.) In this problem, we will examine the form of solutions to the Laplace Equation. The Laplace equation is given as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(a) (4 points) First, suppose we can write the function $u(x, y)$ as $u(x, y) = X(x)Y(y)$. Based on this assumption, produce two separate ordinary differential equations for the functions X and Y. (Hint: an unknown constant k should be involved somewhere in your ODEs.)

(b) (6 points) Discuss the different situations when k takes different values. Write out the general solutions for $X(x)$ and $Y(y)$. Based on the form of the solutions of X and Y, is it possible for a nonconstant seperable solution $u(x, y)$ to be periodic with respect to both the x and y variables?