Welcome to D-Day!

also known as Determinant Day!

as well as Lecture 8

Today: Office Hours 1-3 pm 736 Evans

Friday: Quiz through §3.3

Review of 2×2 determinants.

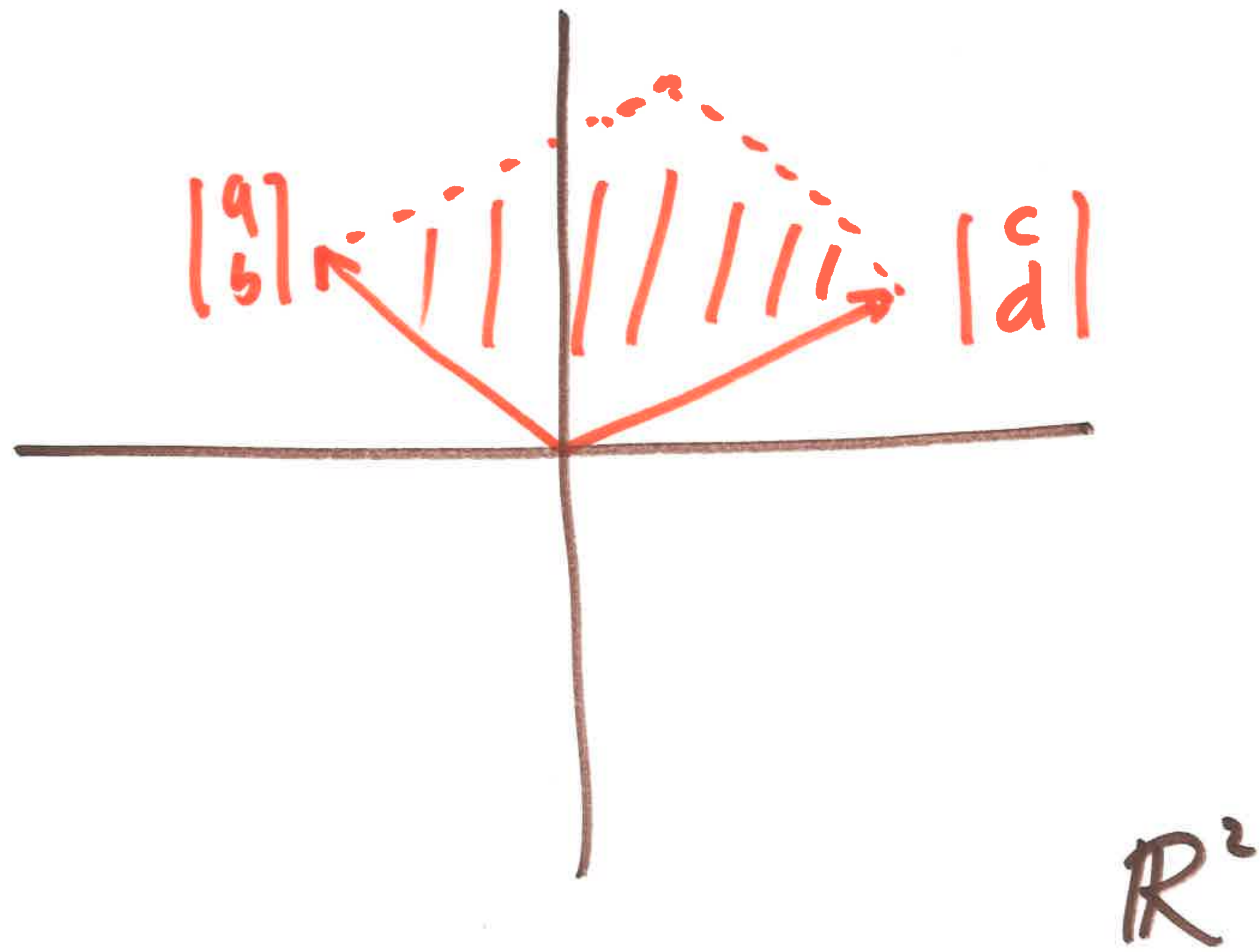
$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $\quad \det(A) = ad - bc$

Theorem (Geometric Interpretation of det)

$|\det(A)|$ = Area of parallelogram with sides the row vectors of A

$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$

Picture



$\begin{bmatrix} a \\ b \end{bmatrix}$
$\begin{bmatrix} c \\ d \end{bmatrix}$
$\mathbb{R}^2$

Why is this true? Let's see how both sides change under row ops

	$\lvert\det(A)\rvert$	Area
(R1) add scale of row to another	unchanged	unchanged
(R2) exchange rows	unchanged (det changes by -1)	unchanged
(R3) scale a row by $\lambda \neq 0$	scaled by $\lvert\lambda\rvert$ (det changes by λ)	scaled by $\lvert\lambda\rvert$

To prove theorem, it suffices now to assume A is in REF.

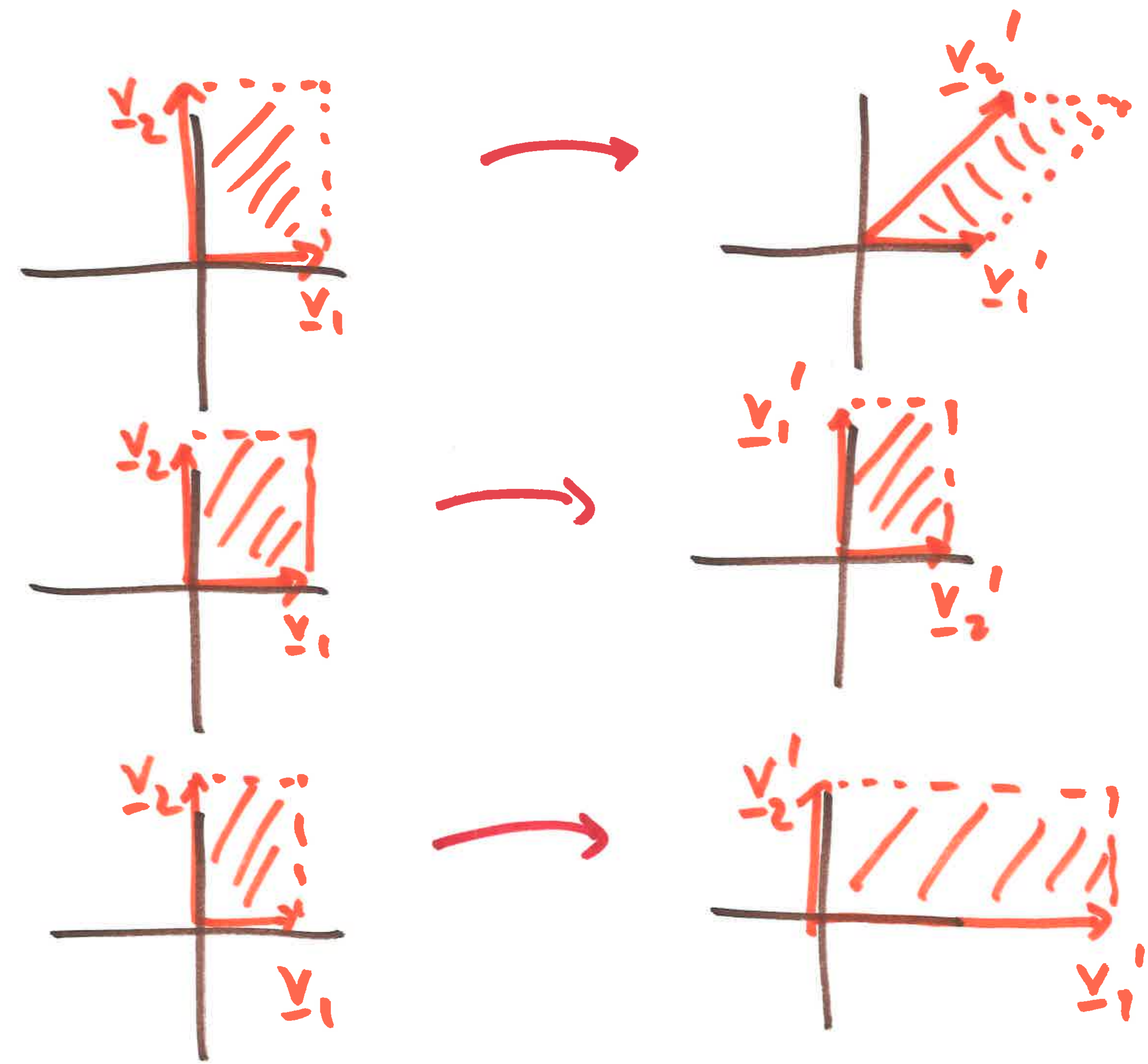
Possibilities	$\lvert\det(A)\rvert$	Area
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0	0
$\begin{bmatrix} a\neq 0 & b \\ 0 & 0 \end{bmatrix}$	0	0
$\begin{bmatrix} 0 & b\neq 0 \\ 0 & 0 \end{bmatrix}$	0	0
$\begin{bmatrix} a\neq 0 & b \\ 0 & d\neq 0 \end{bmatrix}$	$\lvert ad\rvert$	$\lvert ad\rvert$

Picture

(R1)

(R2)

(R3)



<u>Rest of Lecture</u>: for an $n \times n$ matrix, we will introduce its <u>determinant</u> satisfying:

1) similar behavior under row ops

2) similar calculation for matrices in REF

Inductive Definition:

n=1 $\quad A = [a_{11}] \qquad \det(A) = a_{11}$

Suppose we know det of matrices up to size $(n-1)\times(n-1)$

Take $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$ $\quad n \times n$ matrix

$$\text{Set } A_{ij} = \begin{array}{c|c} & \text{col } j \\ \hline \text{row } i & \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ a_{21} & \ddots & & & \\ \vdots & & & & \\ & & & & \\ a_{n1} & & & & a_{nn} \end{bmatrix} \end{array}$$

$(n-1) \times (n-1)$ matrix

$$\det(A) = a_{11} \det(A_{11}) \; \overset{-}{\cancel{+}} \; a_{12} \det(A_{12})$$
$$+ a_{13} \det(A_{13}) - \cdots$$
$$+ (-1)^{1+n} a_{1n} \det(A_{1n})$$

$\boxed{\text{Sum over row 1}}$

<u>Exer</u> Calc. det(A) for following A.

1) $A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$

$$\det(A) = 1\cdot 2 - 2\cdot(-3)$$
$$= 8$$

2) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

$$\det(A) = 1\cdot\det\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$$
$$-2\cdot\det\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$
$$+1\cdot\det\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$$
$$= 1\cdot(-6) - 2\cdot(1) + 1\cdot(3)$$
$$= -5$$

Key properties of determinant

1 Behavior under row ops

$A \xrightarrow[E]{} A'$ E elem. matrix

$A' = EA$

R1 add scale of row to another

$\det(A') = \det(A)$

$\det(E) = 1$

R2 exchange rows

$\det(A') = -\det(A)$

$\det(E) = -1$

R3 scale a row by $\lambda \neq 0$

$\det(A') = \lambda \det(A)$

$\det(E) = \lambda$

2 A in REF

$$A = \begin{bmatrix} \lambda_1 & * & * & \cdots & * \\ & \lambda_2 & * & \cdots & * \\ & & \ddots & & \vdots \\ & 0 & & & * \\ & & & & \lambda_n \end{bmatrix}$$

n pivots

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

$$A = \begin{bmatrix} \lambda_1 & * & \cdots & & * \\ 0 & \lambda_2 & * & \cdots & * \\ & & & & \\ 0 & \cdots & & 0 & 0 \end{bmatrix}$$

$< n$ pivots

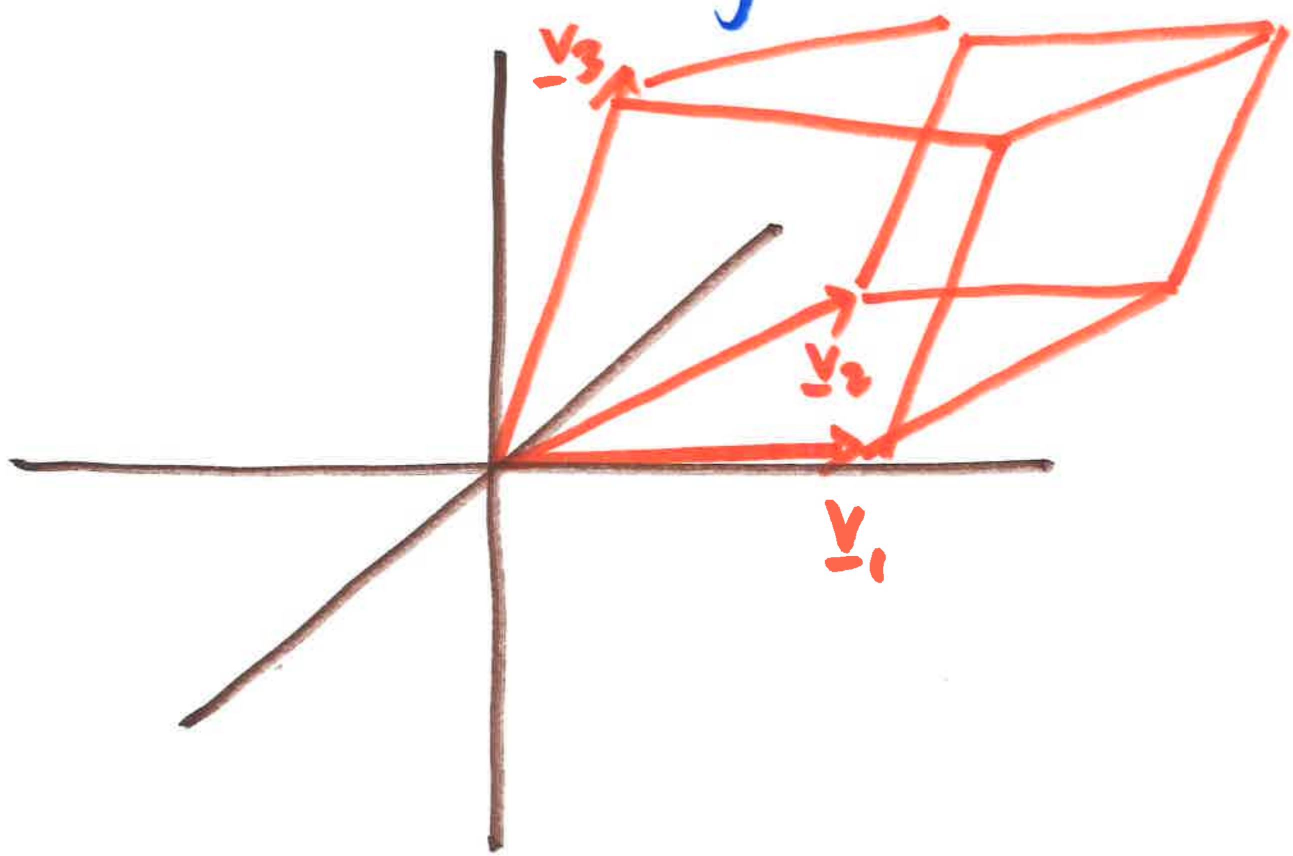
$$\det(A) = 0$$

Observe: If A is upper Δ-ar then $\det(A)$ = product of diag. entries

We'll also see this is true for lower Δ-ar matrices

3 <u>Theorem</u> (Geometric Interpretation)

$|\det(A)| =$ Volume of parallelepiped with edges the rows of A



4 __Theorem__ A invertible $\iff \det(A) \neq 0$

__Proof__ Apply row ops $A \rightsquigarrow A'$ REF

A invertible $\iff A'$ invertible

$\det(A) \neq 0 \iff \det(A') \neq 0$

Thus it suffices to prove the theorem for A in REF.

For A in REF, $\det(A) \neq 0 \iff n$ pivots

$\iff A$ invertible

5 **Theorem** $\det(AB) = \det(A)\det(B)$

$$\det(I_n) = 1$$

In particular: $\det(A^{-1}) = \det(A)^{-1}$ if A^{-1} exists.

Idea of proof: Expand A, B as products of elem. matrices. Then calculate by induction on # of elem matrices appearing

6 Theorem $\det(A^T) = \det(A)$

Nice consequence

rows of A span $\mathbb{R}^n$ & are lin indep

$$\Updownarrow$$

cols of A span $\mathbb{R}^n$ & are lin indep

Remainder of Lecture: formulas involving determinant

① When $n=2,3$, we can expand definition to find: $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A = ad-bc$

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \det A = aei + bfg + cdh - ceg - bdi - afh$$

② Cofactor expansion

A $n \times n$ matrix

Cofactors $C_{ij} = (-1)^{i+j} \det(A_{ij})$

$$A = \begin{array}{cc} & \\ \text{row } i & \begin{bmatrix} a_{11} & a_{12} & & & a_{1n} \\ \vdots & & \ddots & & \vdots \\ & & & & \\ & & & & \\ a_{n1} & & \cdots & & a_{nn} \end{bmatrix} \\ & \text{col } j \end{array}$$

Theorem ((Cofactor expansion)

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(row i expansion)

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(col j expansion)

Note: row 1 expansion = definition of det

Exer Calc $\det(A)$ for

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\det(A) = 1 \cdot C_{32} = (-1)^{3+2} \det(A_{32})$$

$$= (-1)(-3) = 3$$