Welcome to Lecture 5! Intro. to

Matrix Algebra

Friday Quiz through §1.7

Question Authority!

Especially if it's an answer in the back of a textbook...

Before things get more abstract,
let's review where we've been:

Three equivalent concepts:

1) linear system of $m$ eqns in $n$ vars

2) augmented matrix $\underbrace{\left[\begin{array}{c:c} A & \underline{b} \end{array}\right]}_{n+1} \Big\} m$

3) matrix eqn $A\underline{x} = \underline{b}$

## The following are equivalent:

1) There is soln to $A\underline{x} = \underline{b}$.

2) REF of $[A \,\vdots\, \underline{b}]$ has no pivot in augmentation col.

3) $\underline{b} \in \text{Image}(A) = \text{Span}\{\text{cols of } A\}$

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

**The following are equivalent:**

1) There is soln to $A\underline{x}=\underline{b}$ for any $\underline{b}$

2) REF of $A$ has pivot in every row

3) $\mathbb{R}^m = \text{Image}(A) = \text{Span}\{\text{cols of } A\}$

$(\dagger)$

$\boxed{\text{existence}}$

The following are equivalent: (††)

$\boxed{\text{uniqueness}}$

1) If $A\underline{x} = \underline{b}$ has a soln, then soln is unique

2) REF of $A$ has pivot in every col. (no free vars.)

3) $A\underline{x} = \underline{0}$ has only the triv. soln $\underline{x} = \underline{0}$

4) Cols of $A$ are lin indep.

## Why are 1) & 3) equivalent?

Key idea: map given by $A$ is linear!

Suppose $A\underline{x} = \underline{b} = A\underline{y}$ So $A\underline{x} - A\underline{y} = \underline{0}$

By linearity $A\underline{x} - A\underline{y} = A(\underline{x} - \underline{y})$

Conclusion: If 3) holds, then

$\underline{x} - \underline{y} = \underline{0}$ since $A(\underline{x} - \underline{y}) = \underline{0}$

Thus 1) holds: $\underline{x} = \underline{y}$.

Exer. Show

1) $\Rightarrow$ 3)

Continue with <u>lin transf</u> $T: \mathbb{R}^n \to \mathbb{R}^m$

Always represented by unique matrix:

$$A = \begin{bmatrix} | & & | \\ T(\underline{e}_1) & \cdots & T(\underline{e}_n) \\ | & & | \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

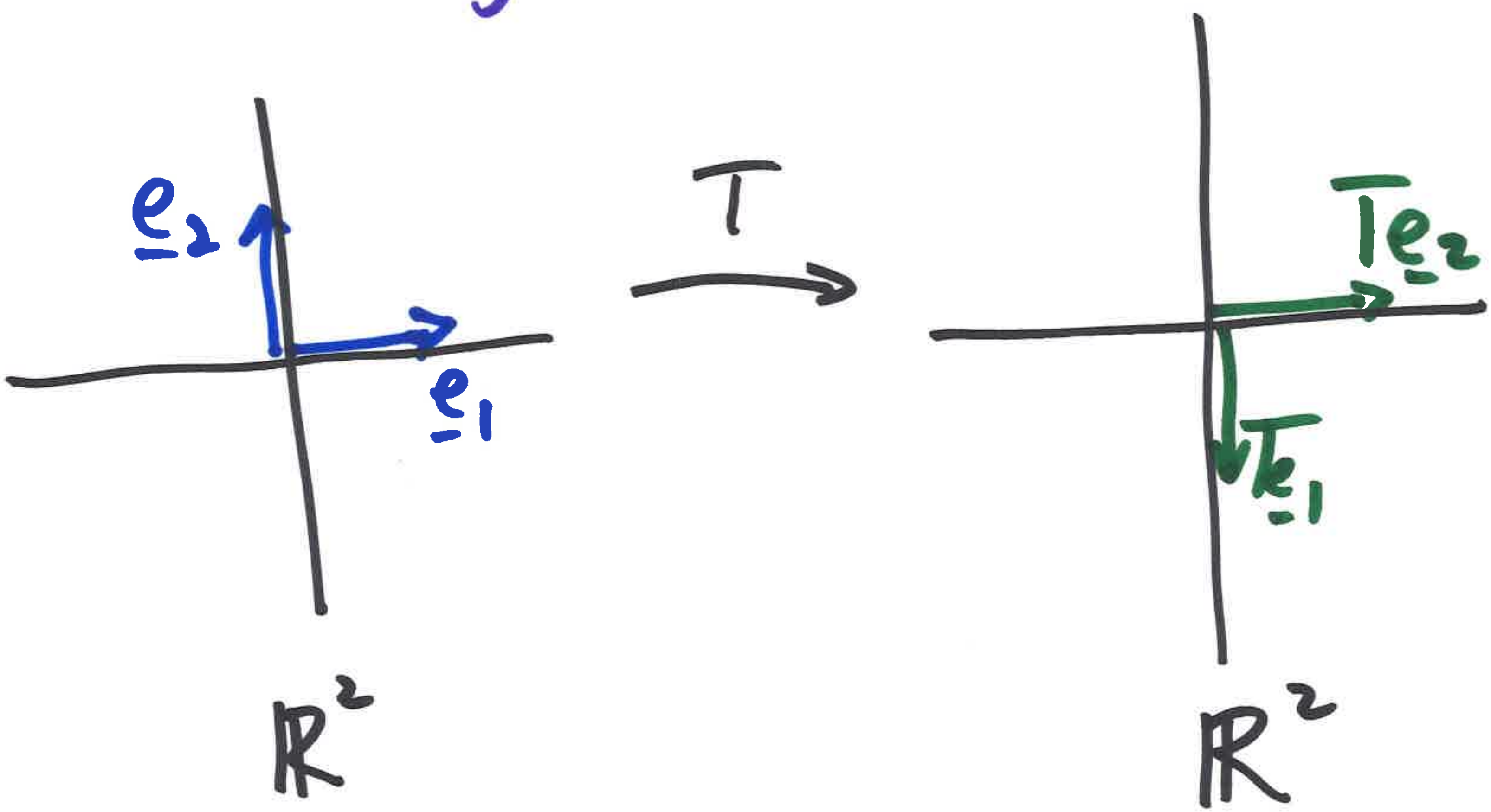
<u>Exer</u> check this!

## Why is this true?

**Key idea:** If $A$ is $m \times n$ matrix and $\underline{v}$ is vector $\underline{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith place

Then $A\underline{v} =$ i-th col of $A$

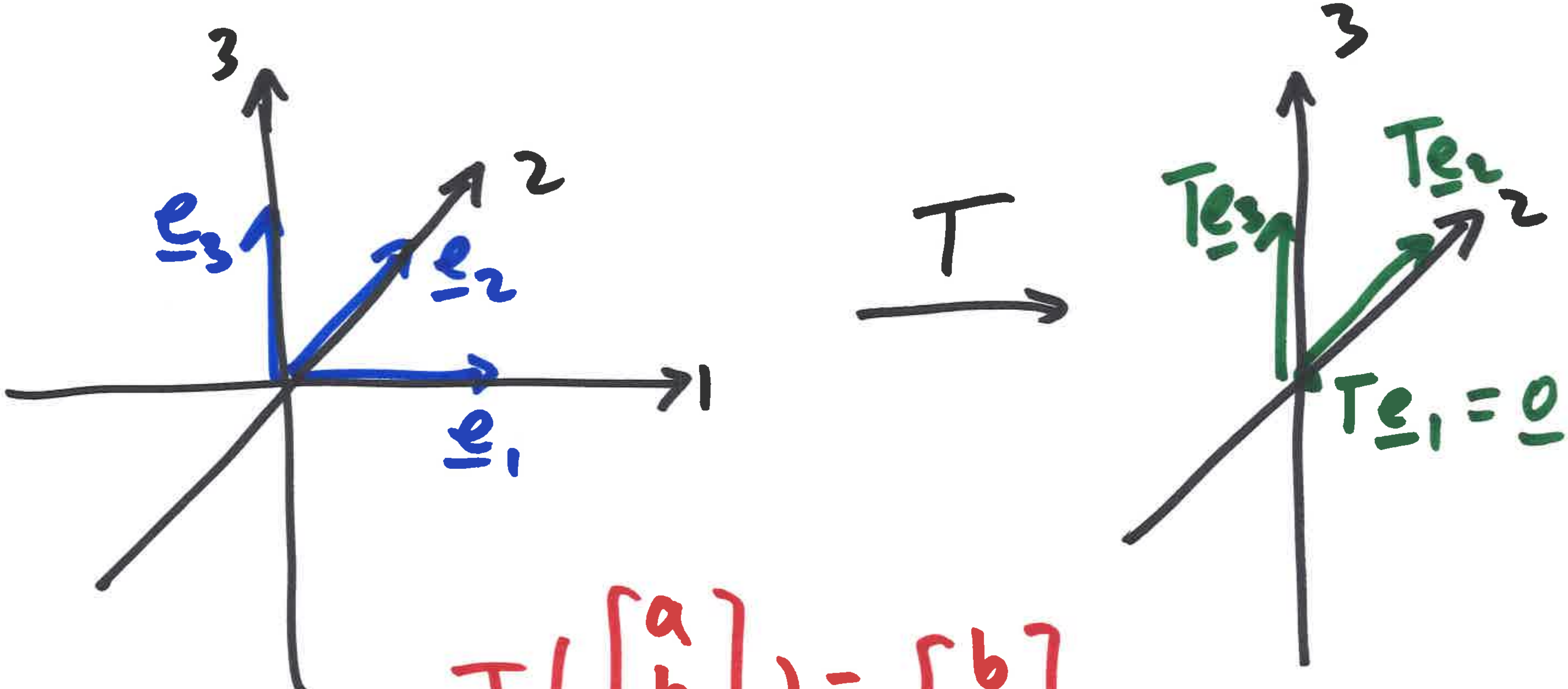
Exer 1) Find ~~lin trans~~ matrix for lin transf $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by $90°$ clockwise.



Matrix of $T$ is

$$A = \begin{bmatrix} | & | \\ T\underline{e}_1 & T\underline{e}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2) Find matrix for lin transf $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by projection onto $2^{nd}$ and $3^{rd}$ coordinates.




$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} b \\ c \end{bmatrix}$$

Matrix of $T$ is

$$A = \begin{bmatrix} | & | & | \\ T\underline{e}_1 & T\underline{e}_2 & T\underline{e}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

__Def__ A lin transf $T : \mathbb{R}^n \to \mathbb{R}^m$ is

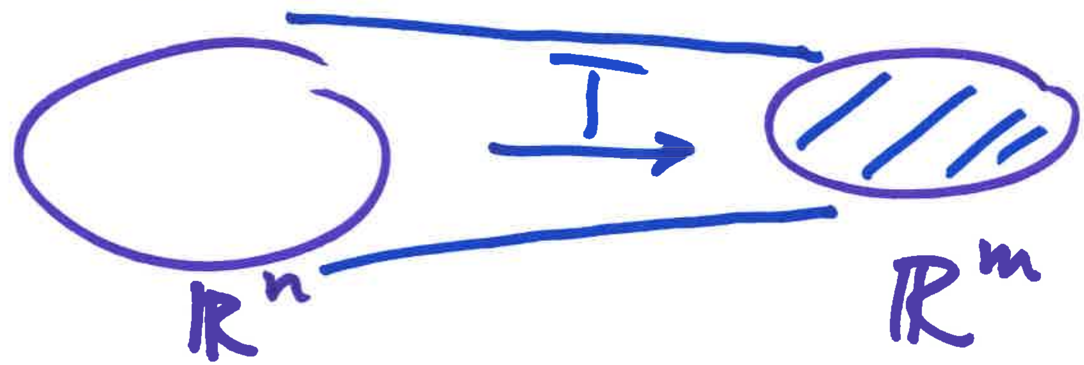
1) __onto__ / __surjective__ if every $\underline{b} \in \mathbb{R}^m$ is image of some $\underline{x} \in \mathbb{R}^n$

$$\underline{b} = T\underline{x}$$

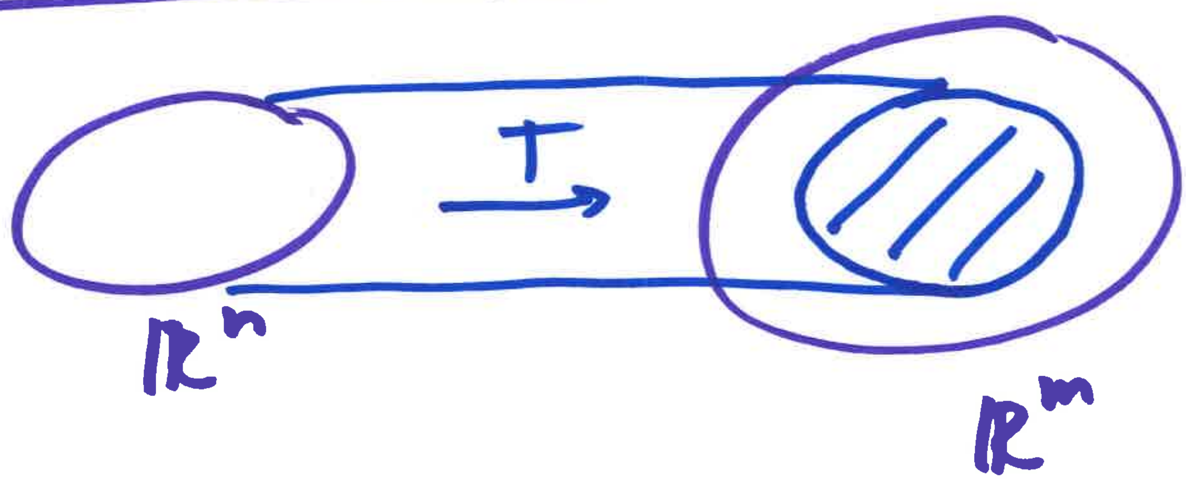
2) __one-to-one__ / __injective__ if whenever $T\underline{x} = T\underline{y}$ for $\underline{x}, \underline{y} \in \mathbb{R}^n$, we have $\underline{x} = \underline{y}$.

Cartoon

1) onto/surjective




2) one-to-one/injective



Exer Is $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ -1 & -1 \end{bmatrix}$$

surj? inj?

For surj: what is Image(A)?

Since $2 < 3$, there cannot be a pivot in each row.

Conclusion: not surj!

For inj: what are solns to $A\underline{x} = \underline{0}$?

$\rightsquigarrow$ REF check there is a pivot in each col.

Conclusion: inj!

A lin transf $T : \mathbb{R}^n \to \mathbb{R}^m$ is

1) surj. $\iff$ (†) holds for its matrix $A$

2) inj $\iff$ (††) holds for its matrix $A$

## Special case when $m=n$

**Theorem** $T: \mathbb{R}^n \to \mathbb{R}^n$ lin transf

Then $T$ is surj $\iff$ $T$ is inj

**Proof:** $T$ is surj $\iff$ REF has pivot in every row

$m=n$!

$\iff$ REF has pivot in each col $\iff$ $T$ is inj.

## Matrix Algebra

What can we do with matrices?

1) **add**: $A, B$ $m \times n$ matrices

$A+B$ $m \times n$ matrix with entries

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

2) **scale**: $A$ $m \times n$ matrix, $c$ number

$cA$ $m \times n$ matrix with entries

$$(cA)_{ij} = cA_{ij}$$

3) <u>multiply</u> $A$ $m \times n$, $B$ $n \times p$

$AB$ $m \times p$ matrix given by

$$AB = \underbrace{\left.\begin{bmatrix} | & & | \\ A\underline{b}_1 & \cdots & A\underline{b}_p \\ | & & | \end{bmatrix}\right\}m}_{p}$$

where $B = \begin{bmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_p \\ | & & | \end{bmatrix}$

# Cartoon of AB

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "B"] \arrow[rr, bend left, "AB"] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

AB is <u>composition</u> of first B followed by A

Exer Calculate $AB$ where

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$4 \times 2$ $\qquad$ $2 \times 3$

$$AB = \begin{bmatrix} 2 & 1 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix} \qquad 4 \times 3$$

Matrix alg has many good properties

... see book!

**Caution:**

1) $AB \neq BA$ ~~not always~~ (usually)

**not commutative**

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

2) $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$$

3) $AB = AC \not\Rightarrow B = C$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$