Lecture 3 Linear combinations, spans, linear independence

Friday: Quiz through §1.3

Next Tuesday: Office Hours 1-3pm
736 Evans

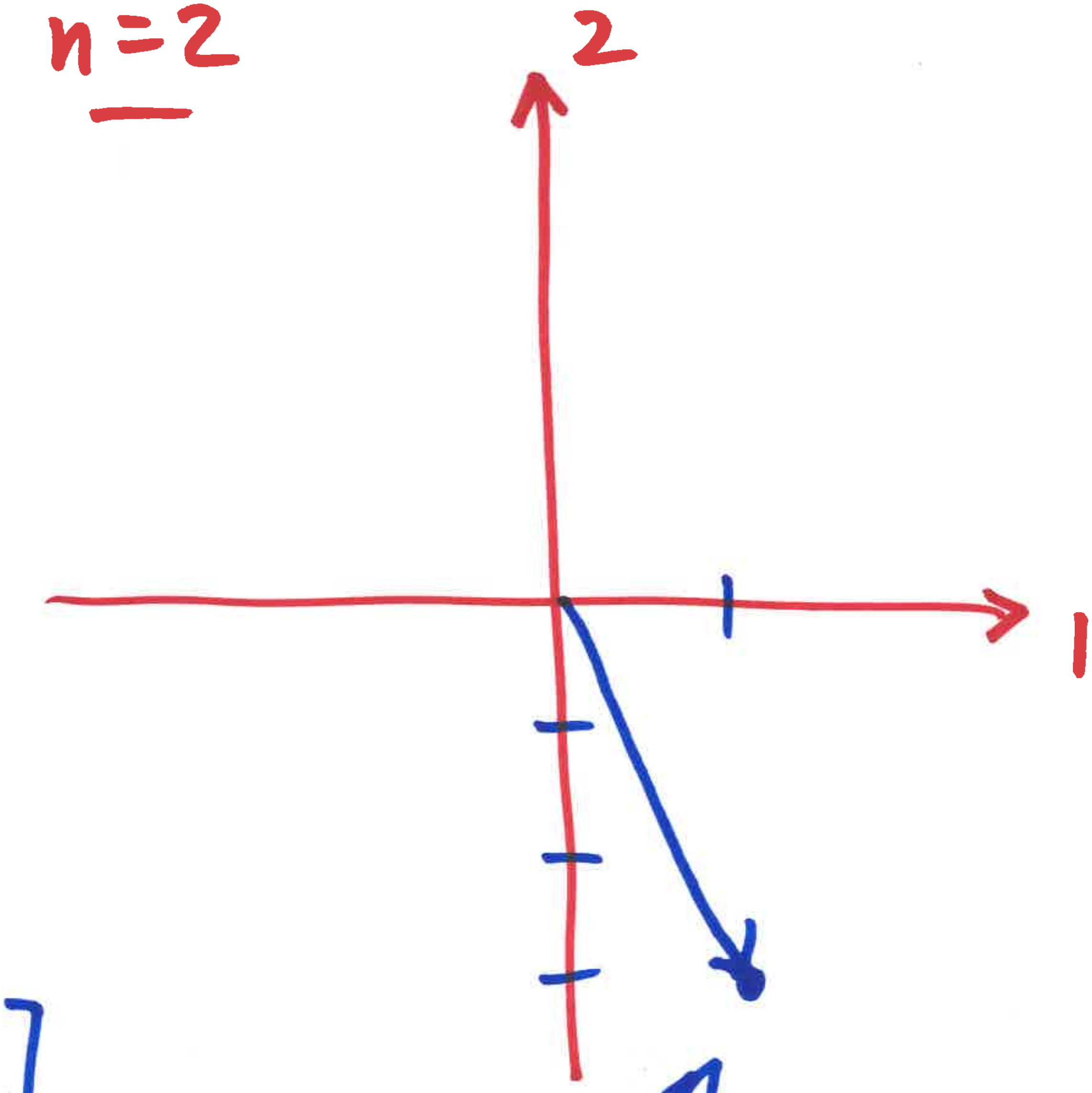
Happy Thursday!

__Theme for today__: lin systs with an emphasis on cols (rather than rows) of avg. matrix

__Def.__ An __n-vector__ is an ordered list of n numbers, drawn as a $n \times 1$ matrix

$$\underline{u} = \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{1} \left.\vphantom{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}\right\} n$$

Example $n=2$

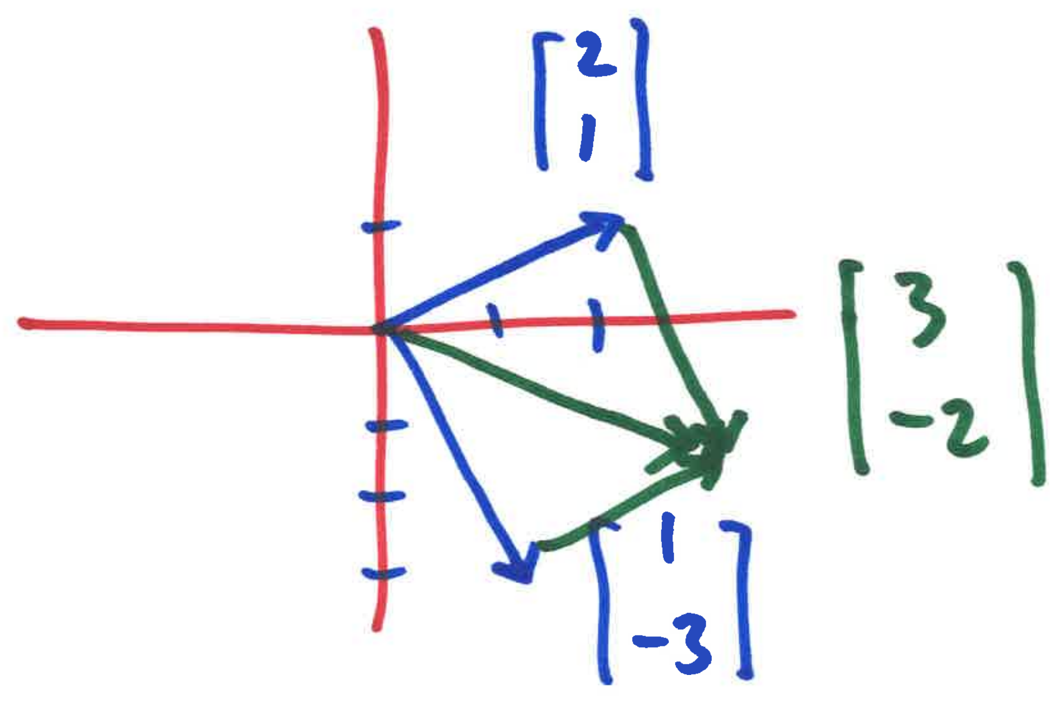


$$\underline{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

What can we do with vectors?

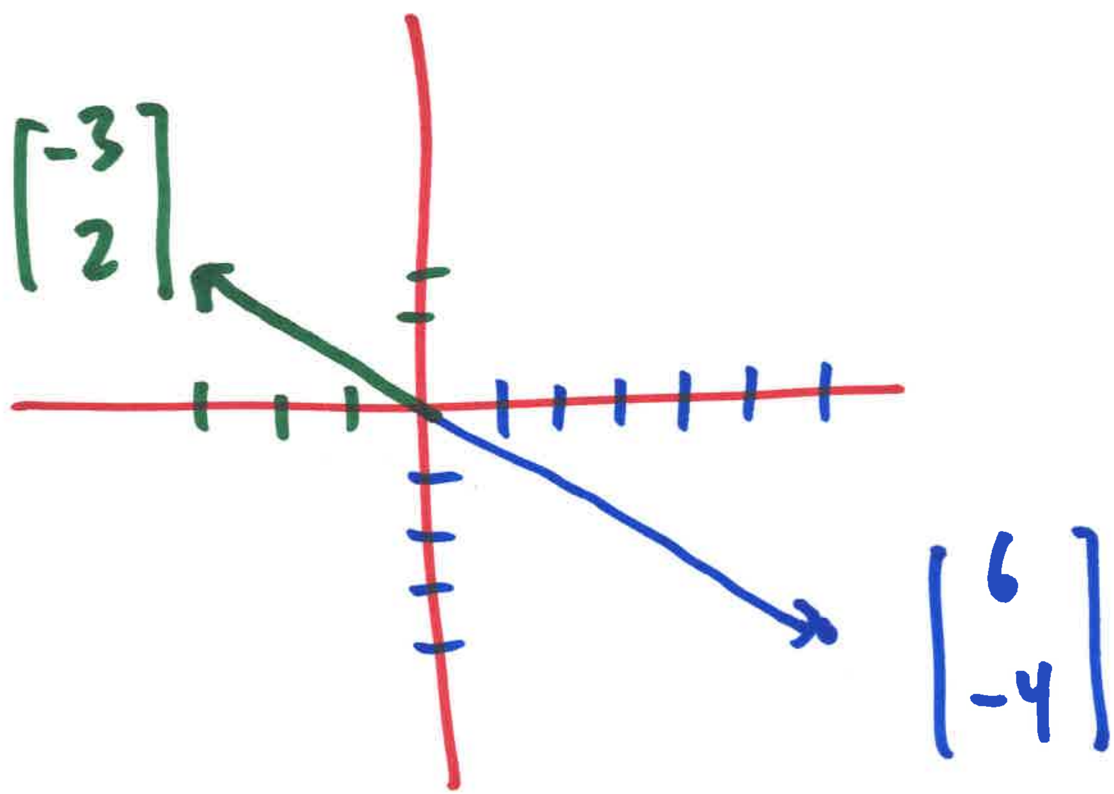
1) Add vectors by adding components

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$



2) Scale vectors by scaling components

$$-\frac{1}{2}\begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$



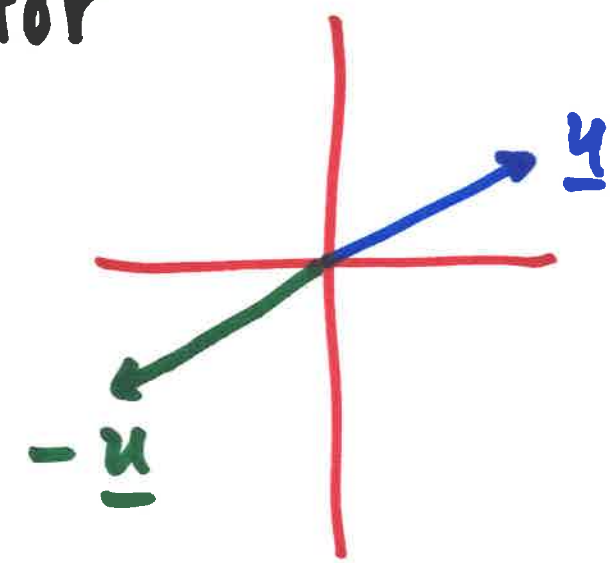
Special vectors

1) zero vector $\underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

2) given vector $\underline{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

opposite/negative vector

$$-\underline{u} = \begin{bmatrix} -a_1 \\ \vdots \\ -a_n \end{bmatrix}$$



Notation We write

$\mathbb{R}^n$ = set of all n-vectors with real components

$\mathbb{C}^n$ = set of all n-vectors with complex components

Caution: some things you cannot do with vectors

1) can not add vectors of different sizes.

2) can not multiply two vectors (though we will talk about inner products soon...)

<u>Def</u> A vector $\underline{u}$ is a <u>linear combination</u> of vectors $\underline{v}_1, \dots, \underline{v}_k$ with coefficients $a_1, \dots, a_k$ if

$$\underline{u} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_k \underline{v}_k$$

__Observe__ $\underline{u}$ is a lin comb. of $\underline{v}_1, \dots, \underline{v}_k$ with coeffs $a_1, \dots, a_k$

$\Updownarrow$ (if and only if)

coeffs $(a_1, \dots, a_k)$ solve the lin syst.

$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k & \underline{u} \\ | & | & & | & | \end{array}\right]$$

__Exer__ For what c is $\underline{u}$ a lin comb of $\underline{v}_1, \underline{v}_2$?

$$\underline{u} = \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix} \qquad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix} \qquad \underline{v}_2 = \begin{bmatrix} c \\ 0 \\ -1 \end{bmatrix}$$

We seek numbers a_1, a_2 so that

$$\underline{u} = a_1 \underline{v}_1 + a_2 \underline{v}_2$$

Equivalenty we seek to solve lin syst

$$\left[\begin{array}{cc|c} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{u} \\ | & | & | \end{array}\right] = \left[\begin{array}{cc|c} 1 & c & 1 \\ 0 & 0 & c \\ c & -1 & 1 \end{array}\right]$$

Scientific method: row reduce to REF...

But example is easy to solve by observing 2^{nd} row $\Rightarrow$ $c = 0$ for there to be soln

Set $c = 0$: $\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{array}\right]$

$\xrightarrow{R2} \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ REF

Take: $x_1 = 1, x_2 = -1$

only soln

Conclusion to exer: c must be 0 for $\underline{u}$ to be lin comb of $\underline{v}_1, \underline{v}_2$

When $c=0$, we found

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

__Def__ The __span__ of a list of vectors $\underline{v}_1, \dots, \underline{v}_k$ is the set of all lin comb:

$$\text{Span}\{\underline{v}_1, \dots, \underline{v}_k\} = \{ a_1\underline{v}_1 + \dots + a_k\underline{v}_k \quad \text{any numbers } a_1, \dots, a_k \}$$

__Note__: If $\underline{v}_1, \dots, \underline{v}_k$ are in $\mathbb{R}^n$, then $\text{span}\{\underline{v}_1, \dots, \underline{v}_k\} \subset \mathbb{R}^n$.

__Exer__ Is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ in $\mathrm{Span}\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\2\\6\end{bmatrix}\right\}$?

We are asking: Is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ a lin comb of $\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\2\\6\end{bmatrix}$?

In other words: Are there coeffs a_1, a_2 so that

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = a_1\begin{bmatrix}2\\1\\0\end{bmatrix} + a_2\begin{bmatrix}-2\\2\\6\end{bmatrix} ?$$

Finally, this is the same as asking: does the lin syst

$$\left[\begin{array}{cc:c} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 0 & 6 & 3 \end{array}\right]$$

have a soln?

Put in REF .... Take $a_1 = 1$
$a_2 = 1/2$

__Exer__ T/F Is it possible for two vectors $\underline{v}_1, \underline{v}_2$ to span all of $\mathbb{R}^3$?

__Question asks:__ given any vector $\underline{u}$ in $\mathbb{R}^3$, are there coeffs a_1, a_2 so that $a_1\underline{v}_1 + a_2\underline{v}_2 = \underline{u}$?

__Equivalently__: can we solve lin syst

$$(*) \quad \left[\begin{array}{cc|c} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{u} \\ | & | & | \end{array}\right] \quad \text{for every } \underline{u} \text{ in } \mathbb{R}^3 \text{ ?}$$

$$\rightsquigarrow \text{RREF} \quad \left[\begin{array}{cc|c} | & | & \\ \underline{w}_1 & \underline{w}_2 & \underline{y} \\ | & | & \end{array}\right] \quad (**)$$

__Note__: can solve $(*)$ for all $\underline{u}$ in $\mathbb{R}^3$

$\iff$ can solve $(**)$ for all $\underline{y}$ in $\mathbb{R}^3$

What are possible RREF?

$$3\left\{\underbrace{\begin{bmatrix} | & | \\ \underline{w}_1 & \underline{w}_2 \\ | & | \end{bmatrix}}_{2}\right.$$ could look like:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Take $\underline{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ So last row is always $[0 \ 0 \,\vdots\, 1]$

Inconsistent: no solns!

Reversing whatever row red. we did gives vector $\underline{u}$ <u>not</u> in span of $\underline{v}_1, \underline{v}_2$!

F

<u>Def</u> List of vectors $\underline{v}_1, \dots, \underline{v}_k$ is

1) <u>linearly dependent</u> if there are coeffs $a_1, \dots, a_k$ not <u>all zero</u> (at least one is nonzer) such that

$$\underline{0} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_k \underline{v}_k$$

2) linearly independent if whenever

$$\underline{0} = a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k$$

it implies $a_1 = \cdots = a_k = 0$

This is the same as not lin dependent.