Lecture 25 Fourier Series, or How I Became a Mathematician

Today Office Hours 12:30-2pm, 736 Evans

Friday Quiz through §10.3

Next week: Reviews during usual lecture times

Final: Wed 12/16 3-6pm Wheeler Aud.

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots$$

Wow!

Welcome Back! Let's briefly review what we've learned about diff eqns before returning to Heat Eqn.

1) nth order linear ODEs $y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y = f$

General Theory

a) homog eqn gen soln $y = c_1 y_1 + \cdots + c_n y_n$

b) nonhomog eqn gen soln $y = y_0 + c_1 y_1 + \cdots + c_n y_n$

c) IVP unique soln with $y(0) = Y_0, \ldots, y^{(n-1)}(0) = Y_{n-1}$

Const coeff eqn can find explicit fns $y_0, y_1, \dots, y_n$

depending on aux eqn and nonhomog term

2) systems of 1st order ODEs $\underline{y}' = A\underline{y} + \underline{f}$

General theory

a) homog eqn : gen soln $\underline{y} = c_1 \underline{y}_1 + \dots + c_n \underline{y}_n$

b) nonhomog eqn : gen soln

$$\underline{y} = \underline{y}_0 + c_1 \underline{y}_1 + \dots + c_n \underline{y}_n$$

c) <u>IVP</u> unique soln with $\underline{y}(0) = \underline{Y}$

<u>Const coeff eqn</u> two methods to find some <u>explicit</u> solns to homog eqn

a) $\underline{y} = e^{rt}\,\underline{u}$, r e-value, $\underline{u}$ e-vector

may not give all solns

b) columns of e^{At}

works always

Relation between 1) nth order ODEs
and 2) 1st order systems of ODEs

$$y \rightsquigarrow \underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} \qquad y_i = y^{(i-1)}$$

converts nth order ODEs
into 1st order systems of ODEs

Back to <u>Heat Eqn</u>

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

<u>IVP</u> : $u(x,0) = f(x)$

<u>BVP</u> : $u(0,t) = 0 = u(L,t)$

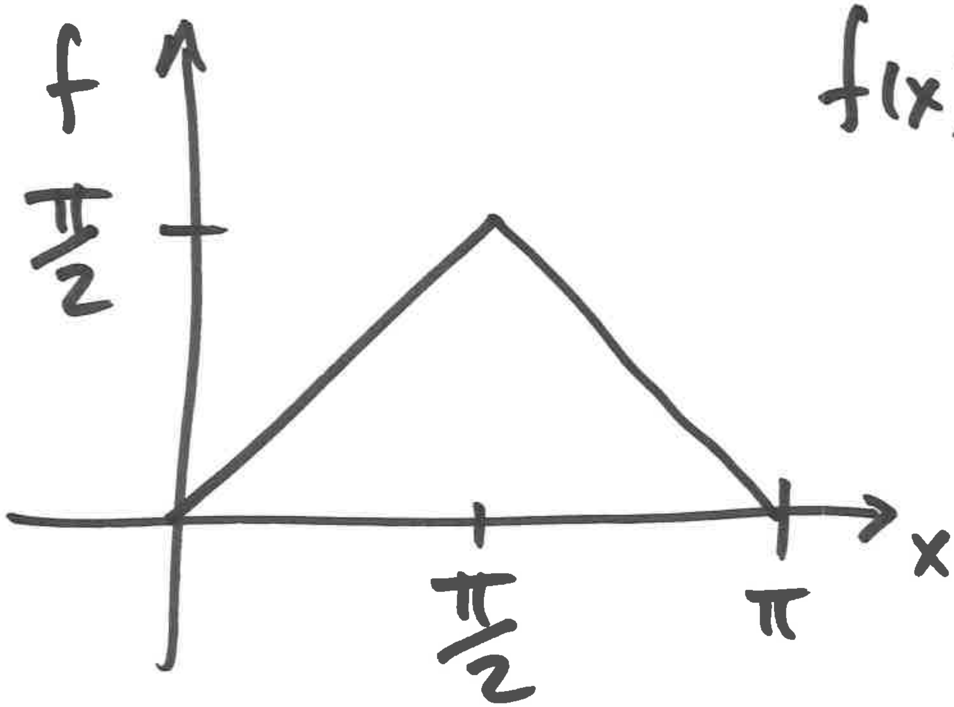
(We also discussed <u>BVP</u>:

$$u(0,t) = u_0$$
$$u(L,t) = u_L \quad)$$

Return to Exer Solve Heat Eqn with

$\beta = 1$, $L = \pi$, and

$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$



By <u>Separation</u> of <u>Variables</u> we found

<u>Gen Soln</u> : $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

Solves Heat Eqn

and BVP

Remaining challenge: solve IVP

$$u(x,0) = \sum_{n=1}^{\infty} c_n u_n(x,0)$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

Want $= f(x)$

need to solve for coeffs

Miraculously Effective Idea: Fourier series!

Proceed as if

$$u_n(x,0) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

form an "orthog. basis"

vect sp $V = \{ f : [0,L] \to \mathbb{R}$ with $f(0) = 0 = f(L) \}$

inner prod $\langle f, g \rangle = \int_0^L f(x) g(x)\, dx$

Outcome will be:

$$c_n = \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} \text{ in eq}^n \quad f = \sum_{n=1}^{\infty} c_n u_n$$

since in an inner prod space V

if $\underline{v}_1, \ldots, \underline{v}_n$ is orthog basis

then $$\underline{v} = \frac{\langle \underline{v}, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1 + \cdots + \frac{\langle \underline{v}, \underline{v}_n \rangle}{\langle \underline{v}_n, \underline{v}_n \rangle} \underline{v}_n$$

Verify orthogonality:

$$\left\langle \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle$$

$$= \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2} \int_0^L \left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right) \right] dx$$

$$= \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

orthogonal!

Confirm they form a "basis"

(not really a basis since we allow "∞-lin combs")

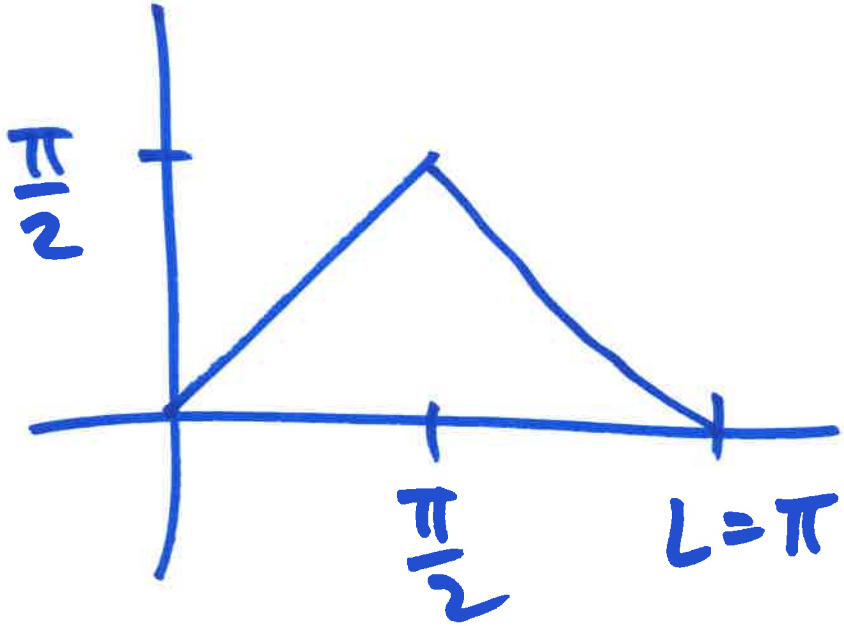
Thm Any fn $f:[0,L]\to\mathbb{R}$ with $f(0)=0=f(L)$ and f' piece-wise continuous is equal to its Fourier series!

$$f(x)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with}$$

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Back to Exer

$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$



solve for coeffs

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

Integration by parts yields:

$$c_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4(-1)^{(n-1)/2}}{\pi n^2} & n \text{ odd} \end{cases}$$

Thm says

$$f(x) = \frac{4}{\pi}\left(\sin(x) - \frac{1}{9}\sin(3x) + \frac{1}{25}\sin(5x) - \cdots \right)$$

Finally soln to Heat Eqn solving
BVP and IVP

$$u(x,t) = \frac{4}{\pi}\left(\sin(x)\,e^{-t} - \frac{1}{9}\sin(3x)\,e^{-9t} + \frac{1}{25}\sin(5x)\,e^{-25t} - \cdots\right)$$

Amazing observation in exercise

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Thm says

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \frac{1}{9}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{25}\sin\left(\frac{5\pi}{2}\right) - \cdots\right)$$

$$= \frac{4}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$$

Whoa!