Lecture 23 Heat and Wave eqns!

Today: Office Hours 2-4pm 891 Evans

Friday: Quiz through §6.2

Next week: Lecture on Tues
then Happy Thanksgiving!
No sections on Wed, Fri

Warmup Find basis of solns of homog eqn

$$\underline{y}' = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \underline{y} \qquad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

constants

fns

Soln **Method 1:** Find e-values r / e-vectors $\underline{u}$

of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

solns will take form

$e^{rt}\,\underline{u}$

single e-value $r = -1$

all e-vectors are scales of $\underline{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\rightsquigarrow$ soln $e^{-1 \cdot t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

This method stalls here... need further ideas.

<u>Method 2</u>: Take cols of e^{At}

Since there is only 1 e-value, we can calculate $e^{At} = e^{(rI + B)t}$

$= e^{rt} e^{Bt}$ $\quad B = A - rI$

Here $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

all higher order terms $= 0$!

$$e^{Bt} = I + Bt + \frac{(Bt)^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Conclusion

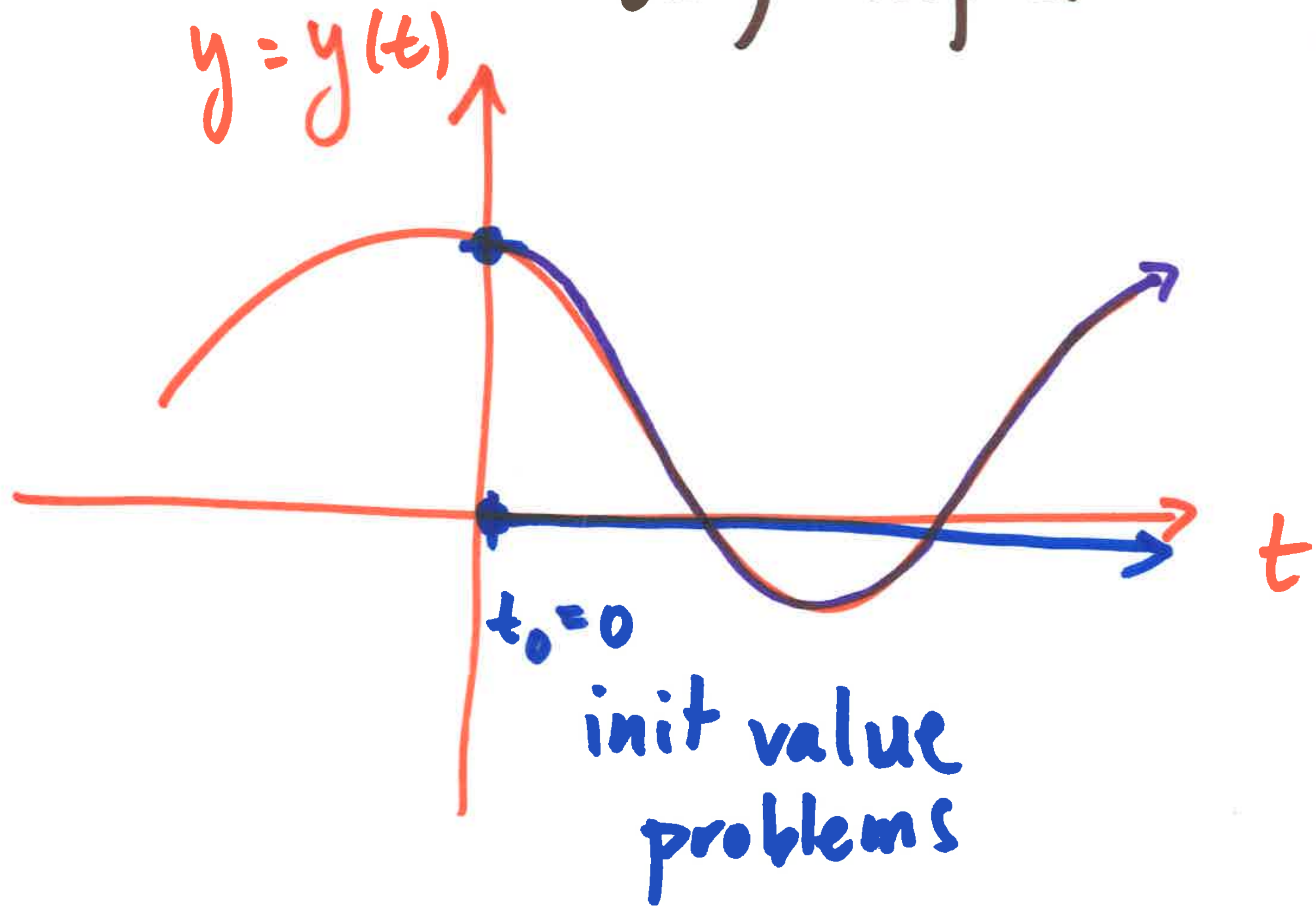
$$e^{At} = e^{-1\cdot t} e^{Bt}$$

$$= e^{-t} \cdot \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

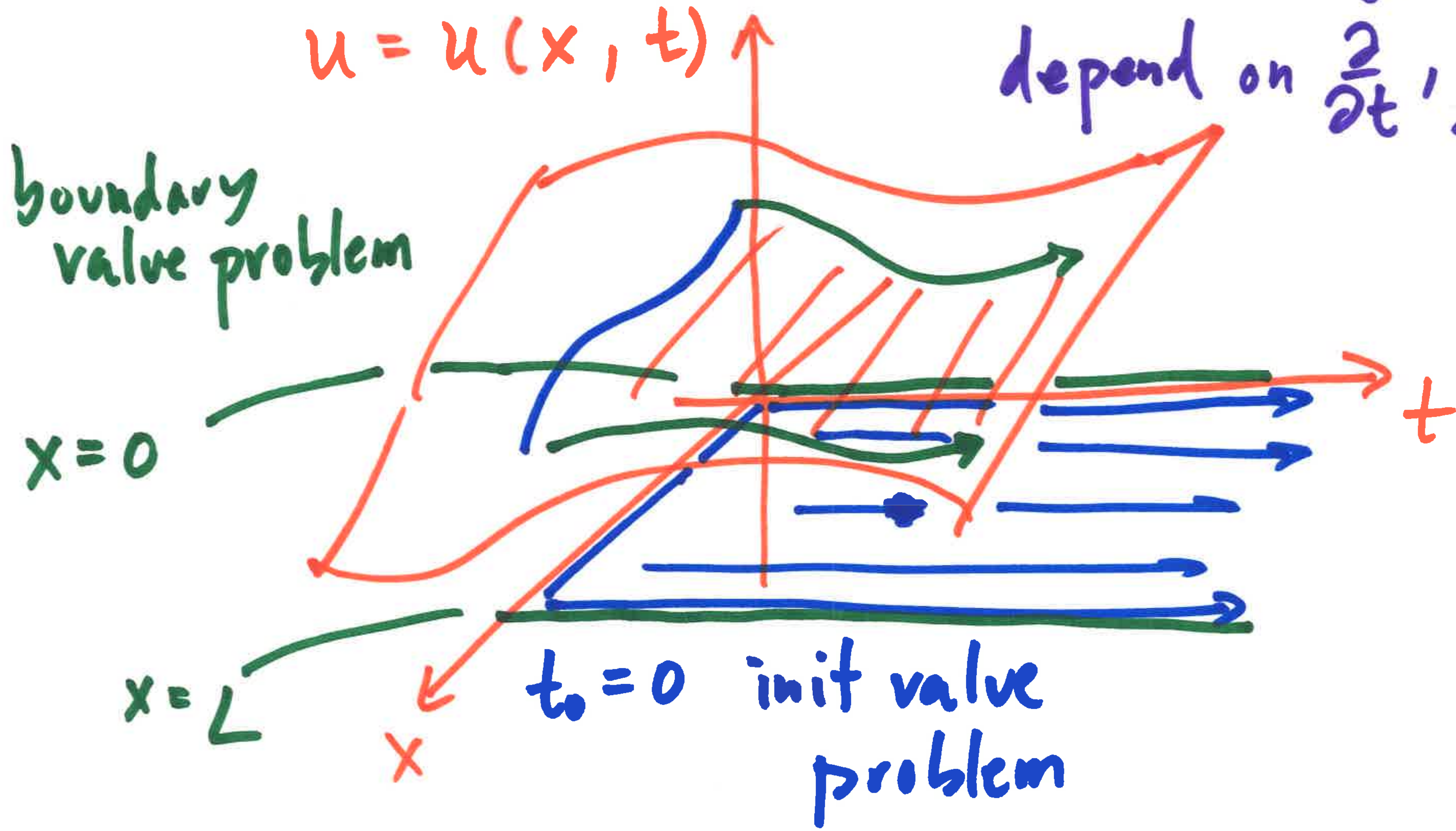
$$= \begin{bmatrix} e^{-t} & e^{-t} t & e^{-t} \frac{t^2}{2} \\ 0 & e^{-t} & e^{-t} t \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Basis of solns: $\begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} e^{-t} t \\ e^{-t} \\ 0 \end{bmatrix}, \begin{bmatrix} e^{-t} \frac{t^2}{2} \\ e^{-t} t \\ e^{-t} \end{bmatrix}$

We've been studying fns of <u>single var t</u> and <u>ODE</u>s = ordinary diff eqns only depend on $\frac{d}{dt}$



Now we'll study fns of two (or more) vars $t, x, \ldots$ and PDEs = partial diff eqns depend on $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \ldots$



Wave eqn

$u = u(x,t)$ displacement of a string

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

α ← const

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}\right\} \text{init values}$$

$$\left.\begin{aligned} u(0,t) &= u_0(t) \\ u(L,t) &= u_L(t) \end{aligned}\right\} \text{bdy values}$$

Heat Eqn

$u = u(x,t)$ temp in a rod

$$\frac{\partial u}{\partial t} = \underset{\text{const}}{\beta} \frac{\partial^2}{\partial x^2} u$$

$u(x,0) = f(x)$ initial value

$$\left.\begin{aligned} u(0,t) &= u_0(t) \\ u(L,t) &= u_L(t) \end{aligned}\right\} \text{boundary values}$$

Exer 1) Find genl sol. of heat eqn indep. of time t : $u = u(x)$

2) Find specific such soln with $u(0) = u_0$, $u(L) = u_L$

Soln Indep of t means $\frac{\partial u}{\partial t} = 0$

Thus we have $0 = \beta \frac{\partial^2}{\partial x^2} u$

Conclude gen soln $u = ax + b$

consts

Bdy <u>value</u> <u>problem</u>: $u(0) = a \cdot 0 + b = u_0$

so $b = u_0$

$u(L) = aL + b = u_L$ so $a = \dfrac{u_L - b}{L}$

$= \dfrac{u_L - u_0}{L}$

<u>Conclude</u> specific soln

$$u = \left(\frac{u_L - u_0}{L}\right) x + u_0$$

<u>Remark</u> This is very useful since:

If we can solve

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

with bdy values

$$u(0,t) = 0 \qquad u(L,t) = 0$$

then we can also solve

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

with bdy values

$$u(0,t) = u_0 \qquad u(L,t) = u_L$$

We take

$$\underbrace{u(x,t)}_{\text{soln with } u(0,t)=0=u(L,t)} + \underbrace{\left(\frac{u_L - u_0}{L}\right)x + u_0}_{\text{soln in exer}}$$

Now let's solve

$$\frac{\partial}{\partial t} u = \beta \frac{\partial^2 u}{\partial x^2}$$

bdy values

with $u(0,t) = 0 = u(L,t)$;

and $u(x,0) = f(x)$!

init values

__Method:__ Separation of Variables

... reduce PDE to ODEs

Seek soln $u(x,t) = X(x)\,T(t)$

$$\frac{\partial u}{\partial t} = X(x)\,T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$$

Heat Eqn $\Rightarrow X(x)\,T'(t) = \beta X''(x)\,T(t)$

Thus $$\underbrace{\frac{1}{\beta}\frac{T'(t)}{T(t)}}_{\text{indep of } x} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{indep of } t}$$

Thus $$\frac{1}{\beta}\frac{T'(t)}{T(t)} = \underbrace{-\lambda}_{\text{const}} = \frac{X''(x)}{X(x)}$$

Decoupled PDE into ODEs!

$$T'(t) + \lambda\beta\, T(t) = 0 \qquad (*)$$

$$X''(x) + \lambda X(x) = 0 \qquad (**)$$

Analyze bdy values:

$$u(x,t) = X(x)\,T(t)$$

$$\begin{array}{ccccc} u(0,t) & = & 0 & = & u(L,t) \\ \| & & & & \| \\ X(0)\,T(t) & & & & X(L)\,T(t) \end{array}$$

To have nontrivial solns,

we impose $\boxed{X(0) = 0 = X(L) \quad (\dagger)}$

Solve (**) with bdy values (†):

Gen sol to (**): $X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$\lambda > 0$

If $\lambda = 0$, $\lambda < 0$ then only triv solns

Specific solns

with (†)

$$X(x) = c \sin\left(\frac{n\pi}{L} x\right)$$

$$n = 1, 2, 3, \ldots$$

$$\left(\sqrt{\lambda} = \frac{n\pi}{L}\right)$$