Lecture 22

Really?!? All Higher Order Diff Eqns are First Order!?!

Offiie Hours this week: Thursday 2-4pm
891 Evans

(no office hours today)

Friday: Quiz through §6.2

Next week: Happy Thanksgiving!

No sections on Wed, no quiz on Fri.
or Fri

__Exer__ Write following __3rd order__ IVP as a __first order__ IVP

$$y''' + 7y' - 3y = \sin(t)$$

$$y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 5$$

__Soln__ Introduce new fns

$$y_1 = y, \quad y_2 = y', \quad y_3 = y''$$

These fns satisfy:

1)

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -7y' + 3y + \sin(t) \\ &= -7y_2 + 3y_1 + \sin(t) \end{aligned}\right\} \text{1st order eqns!}$$

2) $y_1(0) = 1$, $y_2(0) = -2$, $y_3(0) = 5$

Organize as matrix eqn:

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \underline{y}' = \begin{bmatrix} y_2 \\ y_3 \\ -7y_2 + 3y_1 + \sin(t) \end{bmatrix}$$

$$y(0) = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -7 & 0 \end{bmatrix}$$

$$\boxed{\underline{y}' = A\underline{y} + \underline{f}} \qquad \underline{f} = \begin{bmatrix} 0 \\ 0 \\ \sin(t) \end{bmatrix}$$

__Def__ A __normal form__ for a lin syst of diff eqns is an eqn

$$\underline{y}' = A\,\underline{y} + \underline{f}$$

nxn-matrix of fns

n-vectors of fns.

<u>Exer</u> Rewrite the following 2nd order syst in normal form

$$x_1'' + 6\sin(t)\,x_1 - 2x_2' = 0$$

$$x_2'' - e^{3t}x_1 = 0$$

<u>Soln</u> Set $y_1 = x_1,\ y_2 = x_1'$

$y_3 = x_2,\ y_4 = x_2'$

<u>These fns satisfy</u>:

$$y_1' = y_2 \qquad y_2' = -6\sin(t)\, y_1 + 2y_4$$

$$y_3' = y_4 \qquad y_4' = e^{3t}\, y_1$$

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad \underset{\sim}{y}' = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -6\sin(t) & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ e^{3t} & 0 & 0 & 0 \end{bmatrix}}_{A} \underset{\sim}{y}$$

$(\underset{\sim}{f} = \underset{\sim}{0}$ homog$)$

<u>Thm</u> There is a unique soln to IVP

$$y' = A\,\underline{y} + \underline{f} \qquad \underline{y}(t_0) = Y_0$$

where A $n \times n$ matrix of fns on (a,b)

$\underline{f}$ n vector of fns on (a,b)

$t_0 \in (a,b)$

and Y_0 n vector of numbers

Usual analysis: V vect sp of n-tuples of fns

$T : V \to V$ lin transf

$$T = \begin{bmatrix} \frac{d}{dt} & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dt} \end{bmatrix} - A$$

Homog eqn: $Ty = 0$

Nonhomog eqn: $Ty = f$

Usual strategy:

1) Find basis of solns $y_1, \ldots, y_n$ to homog eqn.

2) Find single soln y_0 to nonhomog eqn

3) Gen soln $y = y_0 + c_1 y_1 + \cdots + c_n y_n$

Finally, to solve IVP:

Need: $\underline{y}(t_0) = \underline{y}_0(t_0) + c_1 \underline{y}_1(t_0) + \dots + c_n \underline{y}_n(t_0)$

$= Y_0$

Thus need: $c_1 \underline{y}_1(t_0) + \dots + c_n \underline{y}_n(t_0)$

$= Y_0 - \underline{y}_0(t_0)$

Solve lin syst:

$$\begin{bmatrix} | & & | \\ \underline{y}_1(t_0) & \cdots & \underline{y}_n(t_0) \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} | \\ Y_0 - \underline{y}_0(t_0) \\ | \end{bmatrix}$$

__Wronskian__ $W(t) = \det \begin{bmatrix} | & & | \\ y_1(t) & \cdots & y_n(t) \\ | & & | \end{bmatrix}$

__Fact__: $W(t) \neq 0$ all t $\iff$ $y_1, \ldots, y_n$ lin indep solns of homog eqn

__Exer__ Show if matrix eqn comes from an nth order eqn, then Wronskian is exactly Wronskian of previous lectures.

Now specialize to constant coeff.

$$y' = A\,y + \underline{f}$$

$n \times n$ matrix of numbers

Focus on homog eqn

$$y' = A y$$

Think of this as e-vector/e-value eqn with $\lambda = \frac{d}{dt}$!

Thm If $\underline{u}$ is an e-vector of A with e-value r then $e^{rt}\underline{u}$ is a sol'n to $y' = Ay$.

Proof Check $A(e^{rt}\underline{u}) = e^{rt}A\underline{u}$

$$= e^{rt} \cdot r \cdot \underline{u} = \frac{d}{dt}(e^{rt}\underline{u})!$$

<u>Thm</u> Suppose $\underline{u}_1, \dots, \underline{u}_n$ is a basis of e-vectors of A with e-values $r_1, \dots, r_n$ (for example A is symmetric) then $e^{r_1 t}\underline{u}_1, \dots, e^{r_n t}\underline{u}_n$ is a basis of solns to $\underline{y}' = A\underline{y}$

Exer Find gen soln to $y' = Ay$ where

1) $A = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

Soln e-values $0, 5$

e-vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$y = c_1 \overset{1}{\cancel{e^{0t}}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2) $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

Soln: e-values $1, \quad -1$

e-vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underline{y} = c_1 e^{1 \cdot t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-1 \cdot t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Bold, abstract alternative method:

This will always work even if there is not a basis of e-vectors!

Price to pay ... possibly computationally intensive

<u>Recall</u>: soln to $y' = ry$ is

$$y = e^{rt} c \,, \quad c \text{ number}$$

<u>In general</u>: soln to $\underline{y}' = A\underline{y}$ is

$$\underline{y} = e^{At} \underline{c} \,, \quad \underline{c} \text{ vector of numbers}$$

<u>Def</u>. $$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

<u>Check</u> $$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}(I) + \frac{d}{dt}(At) + \frac{d}{dt}\left(\frac{(At)^2}{2!}\right) + \cdots$$

$$= 0 + A + A^2t + \frac{A^3t^2}{2!} + \cdots$$

$$= A e^{At} \; !$$

so $e^{At}\underline{c}$ solves eqn!

__Exer__ Solve $\underline{y}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{y}$

__Soln__ $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{t^2}{2!} + \cdots$

$$= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

Basis of solns $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$