<u>Lecture 21</u> Higher Order Linear Diff Eqns

<u>Friday</u>: Quiz through §4.5

"Any man who reads too much and uses his own brain too little falls into lazy habits of thinking."

Albert Einstein

<u>Exer</u> Solve IVP

$$y''' - y' = t + \sin(t)$$

$$y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0$$

<u>Soln</u> Analyze homog. eqn

$$y''' - y' = 0$$

$$\underbrace{\left(\left(\frac{d}{dt}\right)^3 - \frac{d}{dt} \right)}_{T} y = 0$$

Aux eqn $r^3 - r = 0$

Factor $r(r-1)(r+1) = 0$

roots: $r = 0, 1, -1$

Homog. eqn factors

$$\frac{d}{dt}\left(\frac{d}{dt} - I\right)\left(\frac{d}{dt} + I\right)y = 0$$

Basis for solns $1 = e^{0t}, e^{t}, e^{-t}$

Back to nonhomog eqn

1) Solve $y''' - y' = t$

<u>Guess</u> $\left.\begin{array}{l} t \xmapsto{T} -1 \\ t^2 \longmapsto -2t \end{array}\right\}$ t is in span

<u>Take</u>: $y = -\frac{1}{2}t^2$

2) Solve $y''' - y' = \sin(t)$

<u>Guess</u> $A\cos(t) + B\sin(t)$

$\xmapsto{T} 2A\sin(t) - 2B\cos(t)$

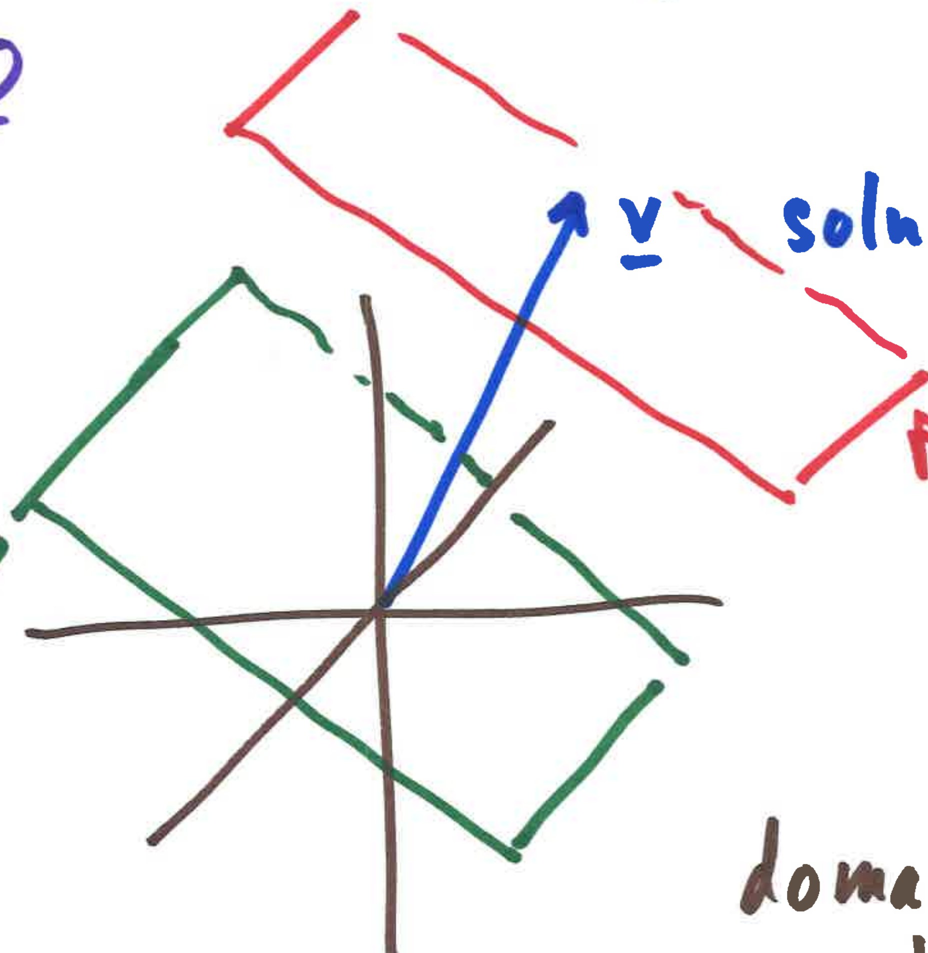
<u>Take</u> $y = \frac{1}{2}\cos(t)$ $\left(A = \frac{1}{2}, B = 0\right)$

<u>Gen soln to nonhomog eqn:</u>

$$y = \underbrace{-\frac{1}{2}t^2 + \frac{1}{2}\cos(t)}_{\text{superposition principle}} + \underbrace{c_1 \cdot 1 + c_2 e^{t} + c_3 e^{-t}}_{\text{gen. soln to homog eqn}}$$

__Reminder__ To solve $T\underline{x} = \underline{b}$, find a soln and then add gen soln to $T\underline{x} = \underline{0}$

__Cartoon__



Now IVP: Evaluate $y(0), y'(0), y''(0)$

Want

$$y(0) = \frac{1}{2} + c_1 + c_2 + c_3 = 2$$

$$y'(0) = c_2 - c_3 = 0$$

$$y''(0) = -\frac{3}{2} + c_2 + c_3 = 0$$

This is a nonhom lin syst for c_1, c_2, c_3!

Organize as matrix eqn

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2-\frac{1}{2} \\ 0 \\ 0-\left(-\frac{3}{2}\right) \end{bmatrix}$$

$\rightsquigarrow$ row reduce and solve

$$c_3 = \frac{3}{4}, \quad c_2 = \frac{3}{4}, \quad c_1 = 0$$

Soln to IVP:

$$y = -\frac{1}{2}t^2 + \frac{1}{2}\cos(t) + \frac{3}{4}e^t + \frac{3}{4}e^{-t}$$

We're done!

General IVP Thm There exists a unique soln to $y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = g$

with $y(x_0) = \gamma_0,\ y^{(1)}(x_0) = \gamma_1,\ \ldots,\ y^{(n-1)}(x_0) = \gamma_{n-1}$

where $p_1, \ldots, p_n, g$ are functions

and $\gamma_0, \ldots, \gamma_{n-1}$ are numbers

Rmk: True on entire line $\mathbb{R}$
or on any open interval (a,b)

<u>Usual analysis</u>: V vect sp of fns

$T : V \to V$ lin transf.

$$T = \left(\frac{d}{dt}\right)^n + p_1\left(\frac{d}{dt}\right)^{n-1} + \cdots + p_n I$$

<u>Homog eqn</u>: $Ty = 0$

<u>Basis of solns</u>: $y_1, \ldots, y_n$

Nonhomog eqn: $Ty = g$

There always is a single soln y_0

Gen soln: $y = y_0 + c_1 y_1 + \cdots + c_n y_n$

Now need to solve IVP

Solve lin syst

$$\begin{bmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & & \vdots \\ y^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \gamma_0 - y_0(x_0) \\ \vdots \\ \gamma_{n-1} - y_0^{(n-1)}(x_0) \end{bmatrix}$$

Why is there always a unique soln?

<u>Def</u> Given fns $y_1, \dots, y_n$, define Wronskian fn

$$W[y_1, \dots, y_n](t) = \det \begin{bmatrix} y_1(t) & \cdots & y_n(t) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t) & & y_n^{(n-1)}(t) \end{bmatrix}$$

invertible at t

$\Longleftrightarrow$

$W[y_1, \dots, y_n](t) \neq 0$

Ex: Calc Wronskian for fns from warmup problem $y_1 = 1$, $y_2 = e^t$, $y_3 = e^{-t}$

Soln $W[y_1, y_2, y_3](t) = \det \begin{bmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{bmatrix}$

$= 2$ in particular $\neq 0$ for any t

__Wronskian Lemma__ The following are equiv.

for fns $y_1, \dots, y_n$ that solve

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$$

1) $W[y_1, \dots, y_n](x_0) \neq 0$ (solve IVP at x_0)
2) $W[y_1, \dots, y_n](t) \neq 0$ any t (solve IVP at any t)
3) $y_1, \dots, y_n$ are lin indep

<u>Exer</u> Can $y_1 = 1$, $y_2 = e^{t^2}$ be solns to $y'' + p_1 y' + p_2 y = 0$?

<u>Soln</u> y_1, y_2 are lin indep.

If they solve $y'' + p_1 y' + p_2 = 0$ then Wronskian Lemma says

$$W[y_1, y_2](t) \neq 0 \text{ any } t$$

$$W[y_1, y_2](t) = \det \begin{bmatrix} 1 & e^{t^2} \\ 0 & 2te^{t^2} \end{bmatrix}$$

$$= 2te^{t^2}$$

$$= 0 \text{ when } t = 0$$

y_1, y_2 cannot both solve 2nd order diff eqn.

Now specialize to constant coeff eqns

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g$$

numbers

(General theory allowed fn coeffs

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = g$$

fns)

<u>Usual analysis</u>: but now can explicitly find solns to homog eqn

Form of solns depends on roots of aux eqn

1) r <u>simple root</u>: e^{rt} solves

2) <u>$r = \alpha + i\beta$ complex root</u>: $e^{(\alpha+i\beta)t}$, $e^{(\alpha-i\beta)t}$

or alternatively $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$ solve

3) repeated root r of <u>mult</u> k $\quad e^{rt}, te^{rt}, \dots, t^{k-1}e^{rt}$ solve

<u>Rmk</u>: if complex root repeats, then can combine cases 2) & 3)

Exer Solve $y^{(4)} - y^{(2)} = 0$

Soln aux eqn $r^4 - r^2 = 0$

factors $r^2(r^2-1) = r^2(r-1)(r+1)$

roots $\underset{\substack{\uparrow \\ \text{mult}=2}}{0}, 1, -1$

Basis of solns $1, t, e^t, e^{-t}$