<u>Lecture 19</u> On to Differential Eqns!

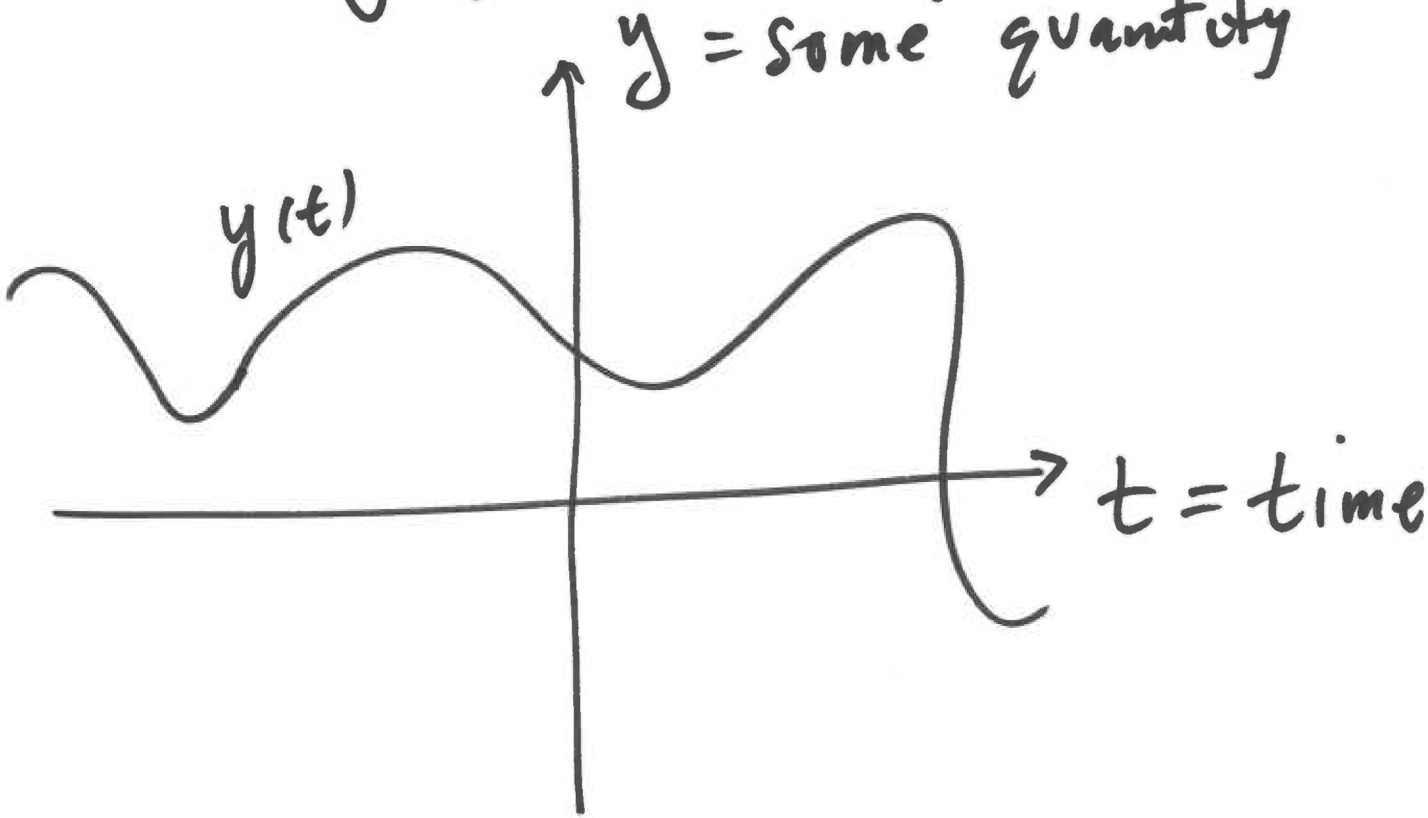
<u>Friday</u> Quiz through § 7.1

"I became an atheist because, as a graduate student studying quantum physics, life seemed to be reducible to second-order differential equations."

Francis Collins

NIH Director

<u>Warmup!</u> Let $y = y(t)$ be a function



Find all solns to $y'' = 2t^2$

Soln Step 1 Find a single soln

$$y = \frac{t^4}{6}$$

Step 2 Find all solns to homog. eqn

$$y'' = 0$$

Gen soln : $y = at + b$, a, b numbers

Gen soln to original inhomog eqn :

$$y = \frac{t^4}{6} + at + b, \quad a, b \text{ numbers}$$

<u>Lin Alg Interpretation</u>

V vect sp of fns on $\mathbb{R}$

$T: V \to V$ lin transf

$T(y) = \left(\frac{d}{dt}\right)^2 (y)$ $\quad \underline{b} = 2t^2$

Find all solns to $T\underline{x} = \underline{b}$.

<u>Step</u> 1 Found single soln $\underline{x} = \frac{t^4}{6}$

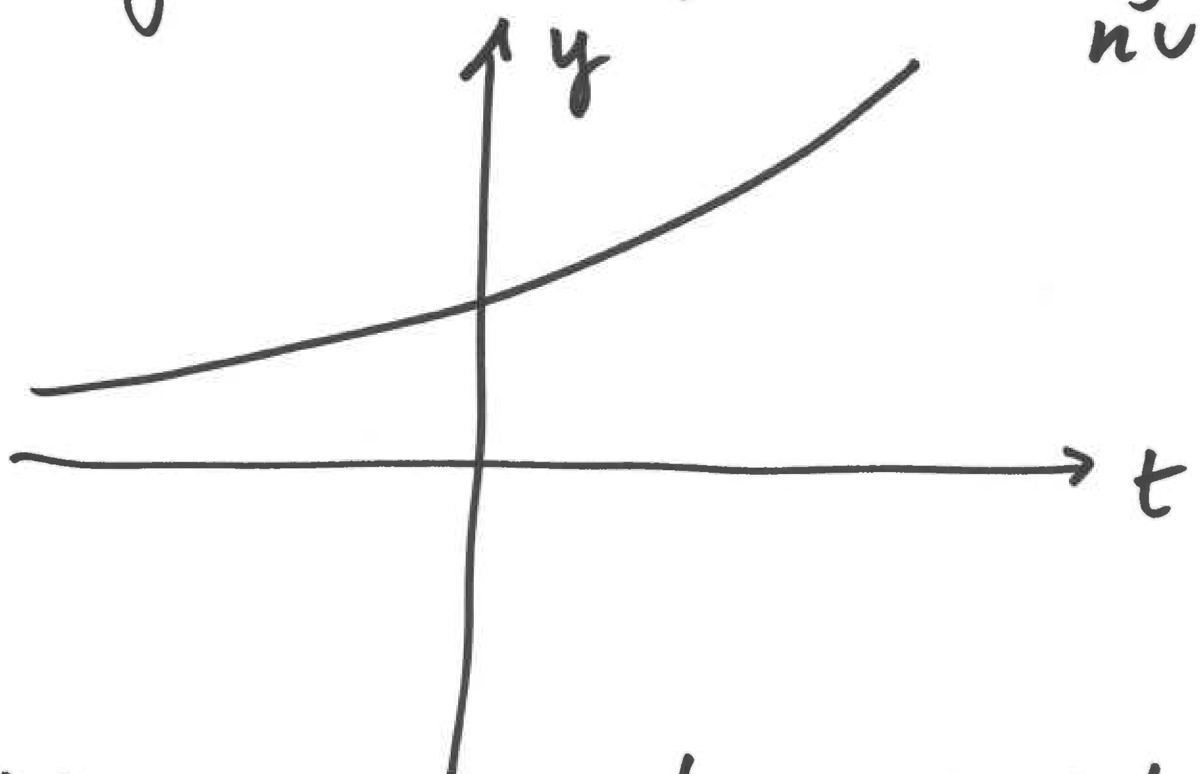
<u>Step</u> 2 Found all solns to $T\underline{x} = \underline{0}$

Basis of $Null(T) = \{1, t\}$

Warmup 2 Find all solns to diff eqn

$$y' = ay, \quad a \text{ any number}$$

Soln $y = ce^{at}$, c any number



Note $y \neq 0$ as long as $c \neq 0$.

__Claim__ $y = ce^{at}$, for c any number, are __all__ solns to $y' = ay$.

__Proof__ Suppose $f' = af$

Calculate $\frac{d}{dt}\left(\frac{f}{e^{at}}\right) = \frac{d}{dt}(fe^{-at})$

$$= f'e^{-at} + f(-a)e^{-at}$$

$$= (af)e^{-at} + f(-a)e^{-at} = 0 \ !$$

So $\frac{f}{e^{at}} = c$ constant. So $f = ce^{at}$.

<u>Lin</u> Alg <u>Interpretation</u> : V vect sp of fns on $\mathbb{R}$

$T : V \longrightarrow V$ lin transf

$T(y) = \frac{d}{dt}(y)$

Solve e-value / e-vector eqn

$$T\underline{x} = \lambda \underline{x}$$

<u>Solns</u> $\lambda = a$ any number (<u>e-value</u>)

e^{at} <u>e-vector</u> for $\underline{\lambda = a}$

A word about our favorite function e^{at}

1) $y = e^{at}$ solves $y' = ay$ and all solns are a scale of it

2) $e^{(a+b)t} = e^{at} \cdot e^{bt}$, a, b any numbers

3) $e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t}$

$$= e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$

Power series expansion

$$e^{at} = 1 + (at) + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

Now on to second order diff eqns!

We know how to solve

$$y' + by = 0$$

Solns: $y = ce^{-bt}$, c constant

New challenge: solve

$$y'' + by' + cy = 0$$

Lin Alg Interpretation V vect sp of fns on $\mathbb{R}$

$T : V \longrightarrow V$ lin transf.

$$T(y) = \left(\left(\frac{d}{dt}\right)^2 + b\,\frac{d}{dt} + c\,I\right)(y)$$

Goal find soln set of $T\underline{x} = \underline{0}$

find basis of $\mathrm{Null}(T)$.

(**Next time**: inhomog version...)

__Useful Def__ __Auxiliary eqn__ where we substitute $r = \frac{d}{dt}$ into lin transf.

$$r^2 + br + c = 0$$

Always __factors__: $(r - r_1)(r - r_2) = 0$

possible roots

1) r_1, r_2 distinct real roots
2) $r_1 = r_2$ repeated real root
3) $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ complex roots

<u>Exer</u> Find all solns to $y'' - 3y' - 4y = 0$

<u>Soln</u> Aux eqn $r^2 - 3r - 4 = 0$

Factor $(r-4)(r+1) = 0 \qquad r_1 = 4,\ r_2 = -1$

(order of roots is unimportant)

Return to lin transf

$$T(y) = \left(\left(\frac{d}{dt}\right)^2 + (-3)\frac{d}{dt} + (-4)I\right)(y)$$

$$= \left(\frac{d}{dt} - 4I\right)\left(\frac{d}{dt} + 1 \cdot I\right)(y)$$

Now easy to solve $T(y) = 0$.

Can individually solve

$$\left(\frac{d}{dt} - 4I\right)(y) = 0$$ soln's $c_1 e^{4t}$, c_1 number

$$\left(\frac{d}{dt} + 1 \cdot I\right)(y) = 0$$ soln's $c_2 e^{-t}$, c_2 number

Gen soln: $y = c_1 e^{4t} + c_2 e^{-t}$

Thm If aux eqn has distinct real roots $r_1 \neq r_2$ then gen soln of

$$y'' + by' + c = 0$$

is of the form

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

c_1, c_2 numbers

Lin Alg Interpretation: $e^{r_1 t}, e^{r_2 t}$ are basis for Null(T) where $T = \left(\frac{d}{dt}\right)^2 + b\frac{d}{dt} + cI$

__Exer__ Solve __Initial Value Problem (IVP)__

$$y'' + 2y' - y = 0$$

$$y(0) = 0, \quad y'(0) = -1$$

__Soln__ __Aux__ eqn $r^2 + 2r - 1 = 0$

__roots__: $r_1 = -1 + \sqrt{2}, \quad r_2 = -1 - \sqrt{2}$

__Gen soln__ $y = c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t}$

Now initial values:

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = (-1+\sqrt{2})c_1 + (-1-\sqrt{2})c_2 = -1$$

Lin system:

$$\begin{bmatrix} 1 & 1 \\ (-1+\sqrt{2}) & (-1-\sqrt{2}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Soln to lin syst: $c_1 = \frac{-\sqrt{2}}{4}$, $c_2 = \frac{\sqrt{2}}{4}$

Conclusion unique soln to IVP

$$y = \left(-\frac{\sqrt{2}}{4}\right) e^{(-1+\sqrt{2})t} + \left(\frac{\sqrt{2}}{4}\right) e^{(-1-\sqrt{2})t}$$

<u>Thm</u> If aux eqn has distinct real roots $r_1 \neq r_2$, then there is a unique soln to the IVP

$$y'' + by' + c = 0$$

$$y(0) = Y_0, \quad y'(0) = Y_1$$

numbers.

__Rmk__ Given gen sol $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to solve IVP it remains to solve the lin syst.

$$\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

invertible since $r_1 \neq r_2$ so $\det \neq 0$.