Lecture 16 Geometry of $\mathbb{R}^n$

... on to Chapter 6!

Friday Quiz through §5.5

Next Tuesday: Review session 12:30-2pm
2040 VLSB

Next Thursday: Midterm 2 through §6.3

<u>Warmup</u> Which of the following matrices are <u>similar</u> to each other?

$$A_1 = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

<u>Reminder</u> $n \times n$ matrices A, B are <u>similar</u> $A \sim B$ if there is an invertible $n \times n$ matrix P so that

$$A = PBP^{-1}$$

Remarks 1) similarity is an equivalence relation

i) $A \sim A$

ii) $A \sim B \Rightarrow B \sim A$

$$A = PBP^{-1} \Longrightarrow P^{-1}AP = B$$

iii) $A \sim B,\ B \sim C \Rightarrow A \sim C$

$$A = PBP^{-1},\ B = QCQ^{-1} \Rightarrow A = P(QCQ^{-1})P^{-1}$$
$$= (PQ)C(PQ)^{-1}$$

2) Interpretation: similarity means:

"there is a basis so that matrix of B with respect to basis is A"

Soln If $A \sim B$ then $\chi_A(t) = \chi_B(t)$

(Careful: converse not true!)

$$\chi_{A_1}(t) = (-2-t)(-t)+1 = t^2+2t+1 = (t+1)^2$$

$$\chi_{A_2}(t) = (3-t)(-t)-2 = t^2-3t+2 = (t-2)(t-1)$$

$$\chi_{A_3}(t) = (-1-t)(4-t)+6 = t^2-3t+2 = (t-2)(t-1)$$

$$\chi_{A_4}(t) = (-1-t)(-1-t)$$

<u>Possibilities</u>: A_1, A_4 may be similar

A_2, A_3 may be similar

But no other pairs.

<u>Observe</u> since A_2, A_3 have <u>distinct</u> e-values so diagonalizable

$$A_2 \sim \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \sim A_3$$

<u>Exer</u> A_1 is <u>not</u> diagonalizable so

$$A_1 \not\sim A_4$$

__Geometry in__ $\mathbb{R}^n$ Up to now, we never discussed __lengths__ of vectors or __angles__ between vectors (though we have discussed area, volume ...)

Def $\underline{u}, \underline{v} \in \mathbb{R}^n$

Dot product / standard inner product / Euclidean inner product

$$\underline{u} \cdot \underline{v} = u_1 v_1 + \cdots + u_n v_n$$

where $\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

Matrix multiplication interpretation:

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = \underbrace{\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{n \times 1}$$

$$= \underbrace{\begin{bmatrix} u_1 v_1 + \cdots + u_n v_n \end{bmatrix}}_{1 \times 1}$$

Key <u>properties</u> of dot product:

1) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

2) $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$

3) $(c\underline{u}) \cdot \underline{v} = c(\underline{u} \cdot \underline{v})$

4) $\underline{u} \cdot \underline{u} \geqslant 0$

$= 0 \iff \underline{u} = \underline{0}$

Def Length of $\underline{u}$

$$\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$$

$$= \sqrt{u_1^2 + \cdots + u_n^2}$$

__Def__ __unit vector__ $\underline{u}$

$$\|\underline{u}\| = 1$$

(unit length)

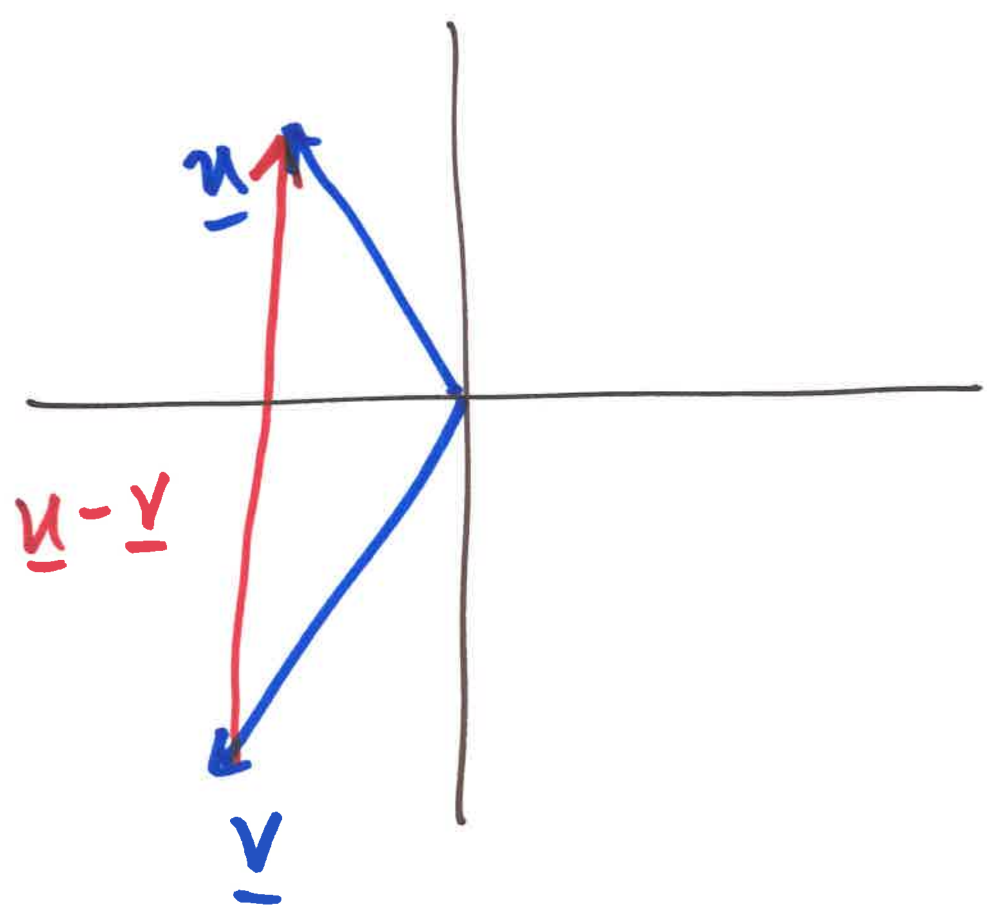
__Normalization__ : $\underline{u} \neq \underline{0}$

unique unit vector in line through $\underline{u}$

$$\hat{\underline{u}} = \frac{1}{\|\underline{u}\|}\underline{u}$$

Def <u>distance</u> between $\underline{u}, \underline{v}$

$$d(\underline{u}, \underline{v}) = \| \underline{u} - \underline{v} \|$$



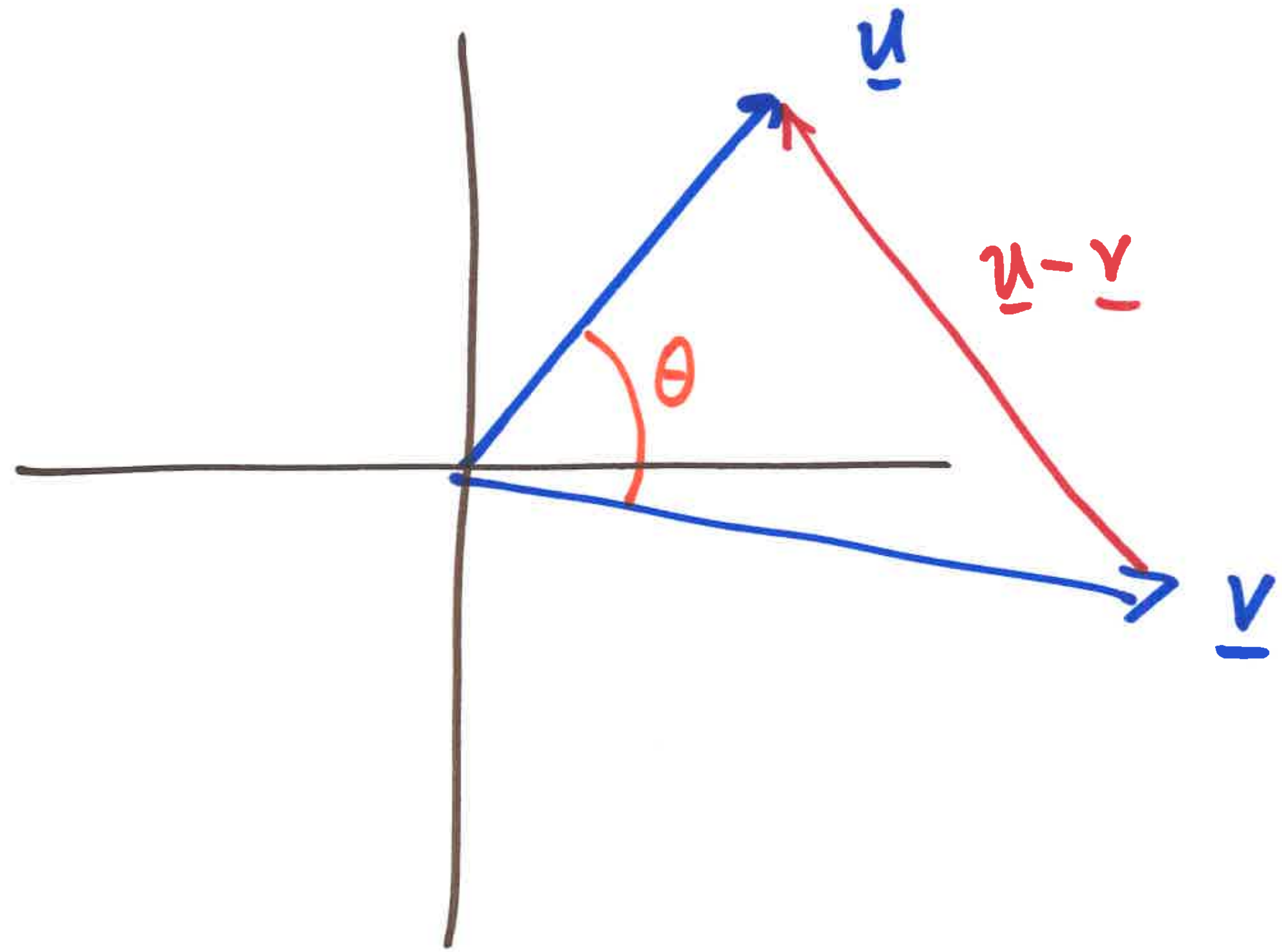
What does $\underline{u} \cdot \underline{v}$ itself mean?

$$\|\underline{u} - \underline{v}\|^2 = (\underline{u} - \underline{v}) \cdot (\underline{u} - \underline{v})$$
$$= \underline{u} \cdot \underline{u} + \underline{v} \cdot \underline{v} - 2\,\underline{u} \cdot \underline{v}$$
$$= \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\,\underline{u} \cdot \underline{v}$$

<u>Law of cosines:</u>

$$\underline{u} \cdot \underline{v} = \|\underline{u}\|\,\|\underline{v}\| \cos\theta$$

<u>Picture</u>

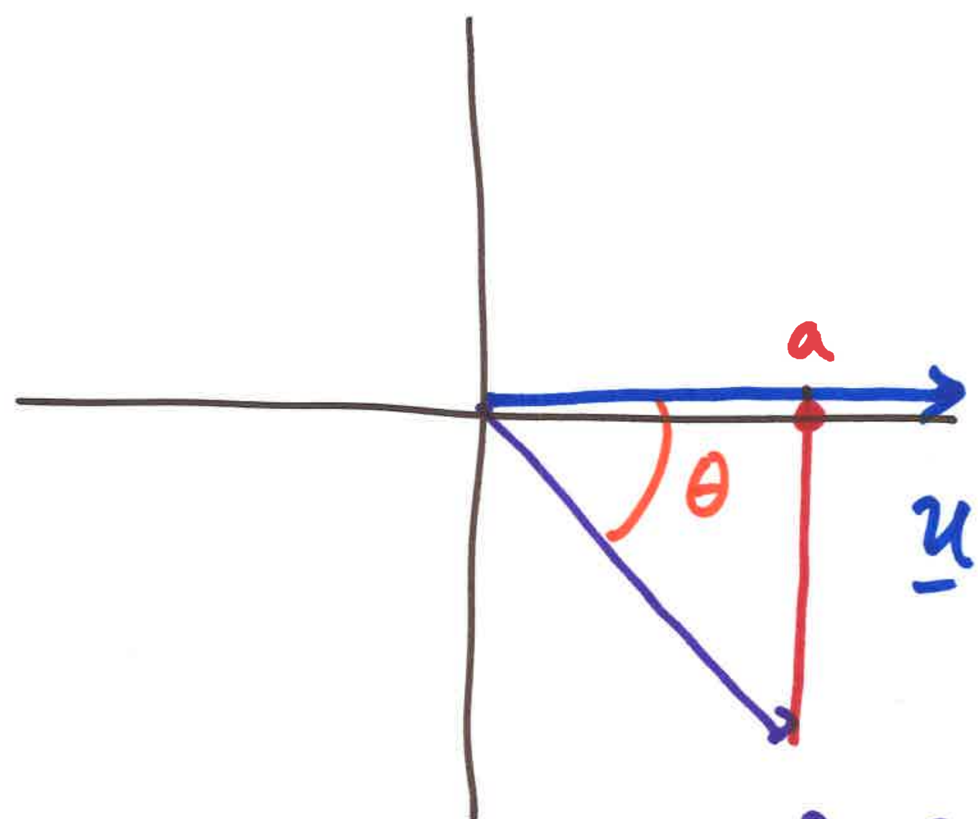


<u>Note</u>: if $\|\underline{u}\| = \|\underline{v}\| = 1$ then

$$\underline{u}\cdot\underline{v} = \cos(\theta) \quad !$$

Why is this true? at least in $\mathbb{R}^2$

inner prod invariant under rotations so can rotate so that $\underline{u}$ points in 1st coord dir



$\underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{u} \cdot \underline{v} = a$

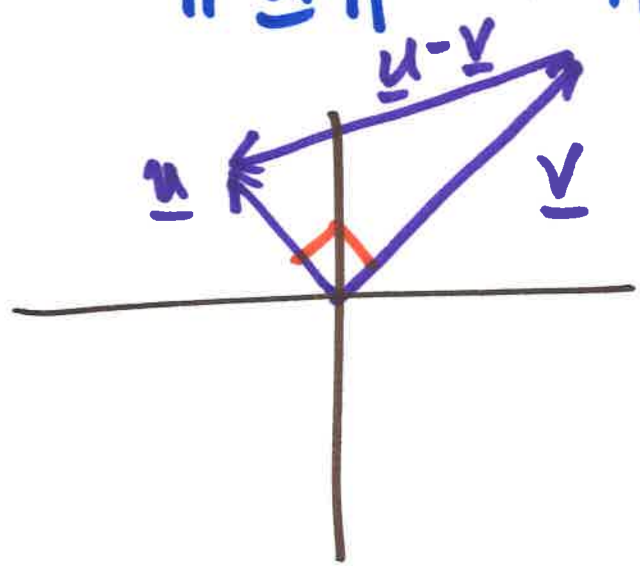
$\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix} \quad a^2 + b^2 = 1$

__Def__ $\underline{u} \perp \underline{v}$ __orthogonal__ means

$$\underline{u} \cdot \underline{v} = 0$$

__Pythag. Thm:__ $\underline{u} \perp \underline{v}$ then

$$\| \underline{u} - \underline{v} \|^2 = \| \underline{u} \|^2 + \| \underline{v} \|^2$$



Def $W \subset \mathbb{R}^n$ subset, orthogonal to W

$$W^{\perp} = \{ \underline{v} \in \mathbb{R}^n \text{ so that } \underline{v} \perp \underline{w} \text{ for all } \underline{w} \in W \}$$

Exer 1) $W^{\perp}$ is a subspace of $\mathbb{R}^n$

2) if W is a subspace, then $(W^{\perp})^{\perp} = W$

3) if $W = \text{span}\{\underline{w}_1, \ldots, \underline{w}_k\}$ then $W^{\perp} = \{\underline{w}_1, \ldots, \underline{w}_k\}^{\perp}$

<u>Exer</u> A $m \times n$ matrix

Find $\mathrm{Col}(A)^\perp$, $\mathrm{Row}(A)^\perp$

<u>Soln</u> Recall $\mathrm{Col}(A) \subset \mathbb{R}^m$, $\mathrm{Row}(A) \subset \mathbb{R}^n$

So $\mathrm{Col}(A)^\perp \subset \mathbb{R}^m$, $\mathrm{Row}(A)^\perp \subset \mathbb{R}^n$

$$A = m\left\{\underbrace{\begin{bmatrix} (a_{ij}) \end{bmatrix}}_{n}\right. \qquad A^T = n\left\{\underbrace{\begin{bmatrix} (a_{ji}) \end{bmatrix}}_{m}\right.$$

Observe:

$$\mathrm{Col}(A)^{\perp} = \mathrm{Row}(A^T)^{\perp} = \mathrm{Null}(A^T)$$

since $\mathrm{Col}(A) = \mathrm{Row}(A^T)$

$A^T \underline{x} = \underline{0}$

$$\mathrm{Row}(A)^{\perp} = \mathrm{Null}(A)$$

<u>Def</u> 1) $\underline{v}_1, \dots, \underline{v}_k$ <u>orthogonal set</u> if

$\underline{v}_i \perp \underline{v}_j$ for all $i \neq j$

2) $\underline{v}_1, \dots, \underline{v}_k$ <u>orthonormal set</u> if

orthogonal set and $\| \underline{v}_i \| = \underline{1}$ all i

3) $\underline{v}_1, \dots, \underline{v}_n$ <u>orthogonal basis</u> if

orthogonal set and basis

4) $\underline{v}_1, \dots, \underline{v}_n$ <u>orthonormal basis</u> if

orthonormal set and basis

<u>Thm</u> $\underline{v}_1, \dots, \underline{v}_k$ orthonormal set

then $\underline{v}_1, \dots, \underline{v}_k$ lin indep.

<u>Proof</u>. Suppose $\underline{0} = a_1 \underline{v}_1 + \dots + a_k \underline{v}_k$

Take dot prod with $\underline{v}_i$:

$$\underline{v}_i \cdot \underline{0} = 0 = a_1 \underline{v}_i \cdot \underline{v}_1 + \dots + a_k \underline{v}_i \cdot \underline{v}_k$$

$$= a_i \cdot \underline{v}_i \cdot \underline{v}_i = a_i$$

Thus all $a_i = 0$ so lin indep.

<u>Exer</u> Is the following set orthog? orthon.?

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \qquad \underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

<u>Soln</u> Check $\underline{v}_i \cdot \underline{v}_j = 0$ all $i \neq j$

so <u>orthog</u>.

But $\|\underline{v}_2\| = 0 \neq 1$

so not <u>orthon</u>.

Exer Is the following an orthog basis?
orthon basis?

$$\underline{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

Soln Check $\underline{v}_i \cdot \underline{v}_j = 0$ all $i \neq j$.
Since all are $\neq \underline{0}$, they must be lin indep (normalize and apply Thm) so basis.
But not orthonormal since $\|\underline{v}_1\| = \sqrt{5}$

Why do we <u>love</u> orthonormal bases?

<u>Exer</u> Find coeffs of $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

with respect to basis of prev. exer.

<u>Traditional method</u>: solve lin syst.

$$A\underline{x} = \underline{v} \quad \text{where} \quad A = \begin{bmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{bmatrix}$$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are coeffs

<u>Soln</u> using the fact that basis is orthog.

Suppose $\underline{v} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3$

<u>Want</u> to find a_1, a_2, a_3

Take dot product with $\underline{v}_i$ to find

$$\underline{v} \cdot \underline{v}_i = a_i \, \underline{v}_i \cdot \underline{v}_i$$

So $a_i = \dfrac{\underline{v} \cdot \underline{v}_i}{\underline{v}_i \cdot \underline{v}_i}$

Nice formula for coeff!

<u>Conclusion</u>

$$\underline{v} = \frac{2}{5}\underline{v}_1 + \frac{1}{3}\underline{v}_2 + \frac{2}{15}\underline{v}_3$$