Good morning and welcome to

Lecture 15 Fun Applications!

Today: Office Hours 1-3pm 736 Evans

Friday: Quiz through § 5.5

Tuesday 10/27: midterm review 12:30-2pm
2040 VLSB

Thursday 10/29: Midterm 2 through § 6.3

Warmup Exer (why diagonalizing is useful in an example)

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Calculate A^{2015}

Soln Last lecture we diagonalized A.

e-values: $7, -4$

e-vectors: $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix}$

So $A = \underset{\varepsilon \leftarrow B}{P} \; D \; \underset{B \leftarrow \varepsilon}{P}$

λ_1

$$D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \qquad \underset{\varepsilon \leftarrow B}{P} = \begin{array}{cc} \underline{v}_1 & \underline{v}_2 \\ \downarrow & \downarrow \end{array} \begin{bmatrix} 1 & -\frac{6}{5} \\ 1 & 1 \end{bmatrix}$$

λ_2

Recall $\underset{B \leftarrow \varepsilon}{P} = \underset{\varepsilon \leftarrow B}{P}^{-1} = \frac{5}{11} \begin{bmatrix} 1 & \frac{6}{5} \\ -1 & 1 \end{bmatrix}$

<u>Observe</u> $A^{2015} = \left(\underset{\mathcal{E} \leftarrow B}{P} \, D \, \underset{\mathcal{E} \leftarrow B}{P}^{-1} \right)^{2015}$

$$= \underset{\mathcal{E} \leftarrow B}{P} \, D^{2015} \, \underset{\mathcal{E} \leftarrow B}{P}^{-1}$$

$$= \begin{bmatrix} 1 & -\frac{6}{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (7)^{2015} & 0 \\ 0 & (-4)^{2015} \end{bmatrix} \left(\frac{5}{11}\right) \begin{bmatrix} 1 & \frac{6}{5} \\ -1 & 1 \end{bmatrix}$$

Conclusion

$$A^{2015} = \left(\frac{5}{11}\right)\begin{bmatrix} 7^{2015} - \left(\frac{-6}{5}\right)(-4)^{2015} & \left(\frac{6}{5}\right)7^{2015} + \left(\frac{-6}{5}\right)(-4)^{2015} \\ 7^{2015} - (-4)^{2015} & \left(\frac{6}{5}\right)7^{2015} + (-4)^{2015} \end{bmatrix}$$

<u>Nice Criterion for diagonalizing</u>:

(sufficient but not necessary)

<u>Thm</u> If A $n \times n$ matrix has k distinct e-values $\lambda_1, \dots, \lambda_k$ then the corresponding e-vectors $\underline{v}_1, \dots, \underline{v}_k$ are lin indep

If $k=n$, then e-vectors $\underline{v}_1, \dots, \underline{v}_n$ form a basis

Proof: By contradiction: suppose not

take $\underline{v}_\ell$ with ℓ smallest such that

$$\underline{v}_\ell = a_1 \underline{v}_1 + \cdots + a_{\ell-1} \underline{v}_{\ell-1} \qquad (*)$$

so $\underline{v}_1, \ldots, \underline{v}_{\ell-1}$ are lin indep.

Apply A to $(*)$

$$A\underline{v}_\ell = \lambda_\ell \underline{v}_\ell = a_1 \lambda_1 \underline{v}_1 + \cdots + a_{\ell-1} \lambda_{\ell-1} \underline{v}_{\ell-1} \qquad (**)$$

Form $\lambda_\ell \cdot (*) - (**)$

$$\underline{0} = a_1(\lambda_\ell - \lambda_1)\underline{v}_1 + \cdots + a_{\ell-1}(\lambda_\ell - \lambda_{\ell-1})\underline{v}_{\ell-1}$$

$\underline{v}_1, \ldots, \underline{v}_{\ell-1}$ lin indep $\Rightarrow$ $a_i(\lambda_\ell - \lambda_i) = 0$ all i.

Case 1 all $a_i = 0$ ↯ then $(*) \Rightarrow \underline{v}_\ell = \underline{0}$ but $\underline{v}_\ell$ is e-vector

Case 2 some $a_i \neq 0$ so $\lambda_\ell - \lambda_i = 0$ ↯ since λ_i all distinct.

<u>Exer</u> $T : P_2 \to P_2$, $T = \frac{d}{dx} + x\frac{d}{dx}$

Is there a basis of P_2 such that matrix $[T]_B$ is diagonal?

<u>Soln</u> In std basis $\mathcal{E} = \{1, x, x^2\}$

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

e-values $= 0, 1, 2$ so T can be diagonalized by Thm

Summary of strategy to diagonalize A

1) Find evalues: find roots of $\chi_A(t)$

2) Find e-spaces: $E_\lambda = \mathrm{Null}(A - \lambda I)$

find basis $\underline{v}_1^\lambda, \ldots, \underline{v}_k^\lambda$

3) Form D and $P_{\mathcal{E} \leftarrow \mathcal{B}}$: $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} | & & | \\ \underline{w}_1 & \cdots\cdots & \underline{w}_n \\ | & & | \end{bmatrix}$$

$\underline{w}_1, \ldots, \underline{w}_n$ list of all e-vectors

$$D = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \; A \; \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$$

$$\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}^{-1}$$

What <u>can</u> go <u>wrong</u>? (There are no bugs in math only features!)

1) $\chi_A(t)$ does not have enough solns

<u>ex</u>: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\chi_A(t) = t^2 + 1$

2) $\dim E_\lambda <$ multiplicity of λ as a root of $\chi_A(t)$

so don't have enough e-vectors

Problem 2) is insurmountable

... Jordan form

What to do about Problem 1) ?

<u>Fundamental Thm of Algebra</u>

Every polynomial $p(t)$ completely factors if we allow complex roots

$$p(t) = c(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$$

λ_i: roots, possibly repeating

Ex $\chi_A(t) = t^2+1 = (t-i)(t-(-i))$

If we accept complex numbers, then we can always fix Problem 1)!

What if we're Neanderthals and don't accept complex numbers?

What's the best we can do with a 2×2 matrix A with complex e-values?

We can find basis $\mathcal{B} = \{\underline{w}_1, \underline{w}_2\}$ so that

$$C = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \, A \, \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$$

with $C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

rotation

$$= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

scaling

$r = \sqrt{a^2+b^2}$, $a = r\cos\theta$, $b = r\sin\theta$

How to find $B = \{\underline{w}_1, \underline{w}_2\}$, C?

Find complex e-values λ_+, λ_-

and complex e-vectors $\underline{v}_+, \underline{v}_-$

Observe $\lambda_+ = a + ib$, $\lambda_- = a - ib$

~~Hear~~

Here are a, b!

$\underline{w}_1 = \mathrm{Re}(\underline{v}_+)$, $\underline{w}_2 = \mathrm{Im}(\underline{v}_+)$

Application of e-values/e-vectors:

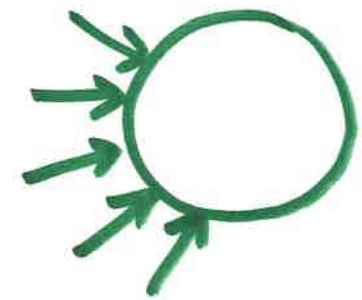
Google's PageRank algorithm

Problem to solve: ranking webpages

Old school: rank by occurences of keywords.

<u>New idea</u>: rank by <u>importance</u>

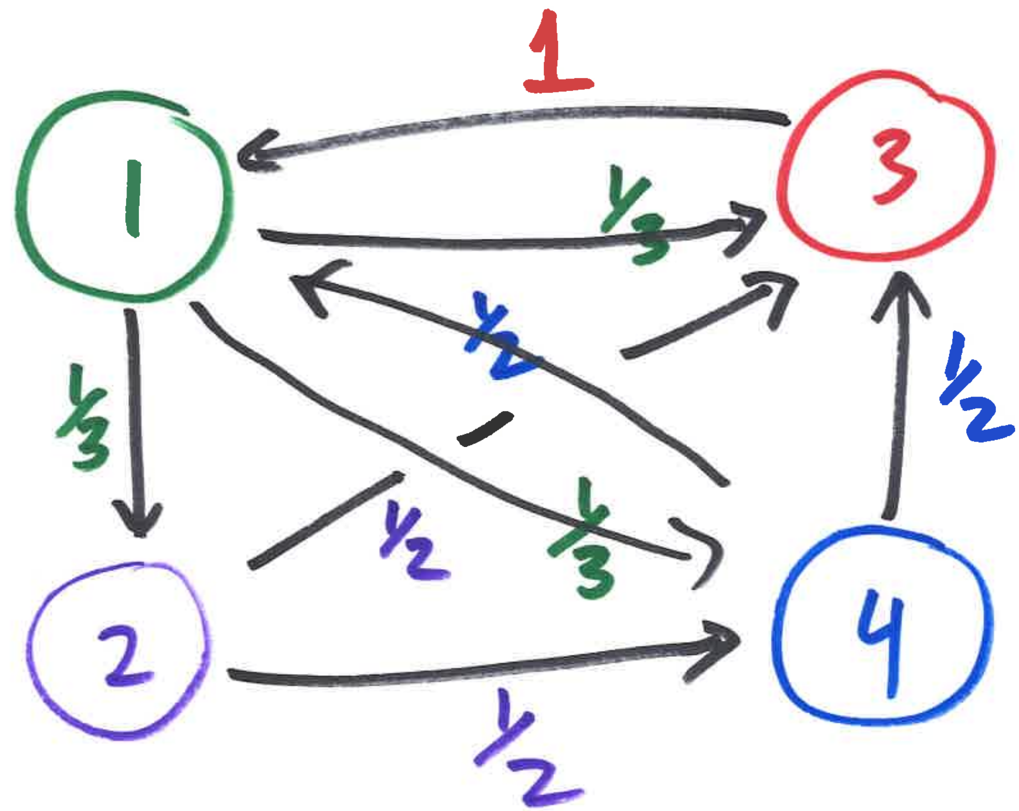
function of 1) <u>popularity</u>



2) <u>authority</u>: importance of those linking

If we have an internet I and a webpage $i \in I$, we'll write x_i for its <u>importance</u>

Mini-Internet I



Organize weights into matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Organize importance into vector

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Solve eqn:

$$A\underline{x} = \underline{x} \quad !$$

"Importance is weighted sum of importances of everyone linking to you."

In <u>example</u>: e-vector with e-value $\lambda = 1$

$$\underline{x} = \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$$

<u>PageRank</u>: 1, 3, 4, 2