Good morning and welcome to Lecture 12!

Bases continued

"A point of view can be a dangerous luxury when substituted for insight and understanding."

— Marshall McLuhan

<u>Fri</u>day: Quiz through § 4.7

<u>Warmup</u> <u>Definition</u> If $T: V \to W$ lin tr. then $\operatorname{rank}(T) = \dim \operatorname{Image}(T)$

Find $\operatorname{rank}(T)$, $\dim \operatorname{Null}(T)$ for

$$T: \mathcal{P}_2 \longrightarrow \mathbb{R}^3$$

$$T(p) = \begin{bmatrix} p(0) \\ \frac{dp}{dx}(0) \\ 4p(1) - p(2) \end{bmatrix}$$

<u>Soln</u> If $p(x) = a_0 + a_1 x + a_2 x^2$ then

$$T(p) = \begin{bmatrix} a_0 \\ a_1 \\ 4(a_0 + a_1 + a_2) - (a_0 + 2a_1 + 4a_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_0 \\ a_1 \\ 3a_0 + 2a_1 \end{bmatrix}$$

<u>Strategy</u> Choose bases of P_2, $\mathbb{R}^3$

Express T as a matrix A

Calculate $\operatorname{rank}(A)$, $\dim \operatorname{Null}(A)$

Take $\mathcal{B} = \{1, x, x^2\}$ basis of P_2

$$C = \left\{ \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$

Then $A = \begin{bmatrix} | & | & | \\ [T(1)]_C & [T(x)]_C & [T(x^2)]_C \\ | & | & | \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank$(A) = 2$

dim Null$(A) = 1$

Theorem Suppose V, W are finite dimensional and $T: V \to W$ lin transf

Then $\dim V = \operatorname{rank}(T) + \dim \operatorname{Null}(T)$

Idea of Proof: Choose bases to obtain matrix and apply old Rank Thm for matrices.

Bases are very useful, how do we construct them?

<u>Exer</u> Extend $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ to a basis of $\mathbb{R}^4$.

<u>Note</u>: list is lin indep

<u>Soln</u> Is $\operatorname{span}\{\underline{v}_1, \underline{v}_2\} = \mathbb{R}^4$?

If yes, done since lin indep.

If no, choose $\underline{v}_3 \notin \operatorname{span}\{\underline{v}_1, \underline{v}_2\}$

For example, take $\underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Now repeat: Is $\text{span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\} = \mathbb{R}^4$?

If yes, done since $\underline{v}_1, \underline{v}_2, \underline{v}_3$ lin indep

If no, choose $\underline{v}_4 \notin \text{span}\{\underline{v}_1, \underline{v}_2 \underline{v}_3\}$

For example, take $\underline{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Now done since $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ lin indep and $\dim \mathbb{R}^4 = 4$ so they span.

Exer Find a subset of vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \underline{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

that is a basis of $\mathbb{R}^3$.

Note: list spans

Soln $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ lin dep since $\dim \mathbb{R}^3 = 3$

Find first $\underline{v}_i \in \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_{i-1}\}$ then throw it out.

Here $i=3$ and $\underline{v}_3 \in \text{span}\{\underline{v}_1, \underline{v}_2\}$

New list: $\underline{v}_1, \underline{v}_2, \underline{v}_4$

Repeat if necessary but here we have 3 vectors that span $\mathbb{R}^3$ so they must also be lin indep.

<u>Theorem</u> (constructing bases)

1) If $\underline{v}_1, \dots, \underline{v}_k$ are lin indep in V and $\dim V = n$ then we can extend to a basis $\underline{v}_1, \dots, \underline{v}_k, \underline{v}_{k+1}, \dots, \underline{v}_n$

2) If $\underline{v}_1, \dots, \underline{v}_k$ span V then some subset of them is a basis

<u>Idea of proof</u>: algorithms of previous exercises.

<u>Rest of lecture</u> What happens if you and I work with different bases? How to compare?

<u>Example</u>: Suppose we have two bases of $\mathbb{R}^n$, how do we compare coords of vectors wrt two bases?

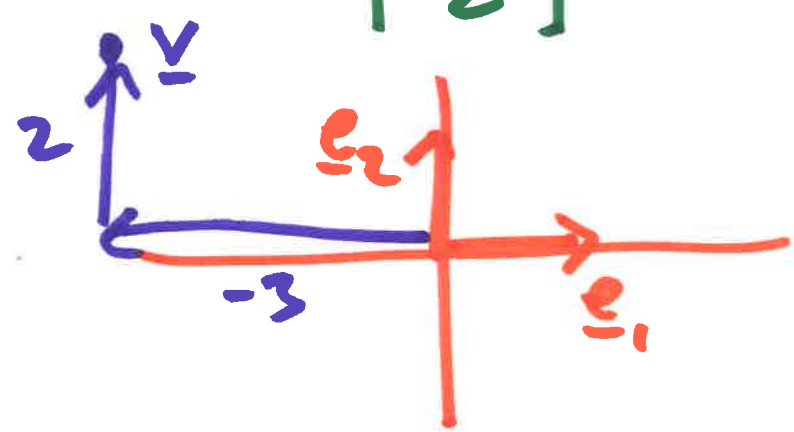
Exer Find coords of $\underline{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$

wrt bases:

1) $\mathcal{E} = \left\{ \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ std basis

2) $\mathcal{B} = \left\{ \underline{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \underline{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Soln 1) $[\underline{v}]_{\mathcal{E}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



2) $P_B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$P_B\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = a_1 \underline{b}_1 + a_2 \underline{b}_2$$

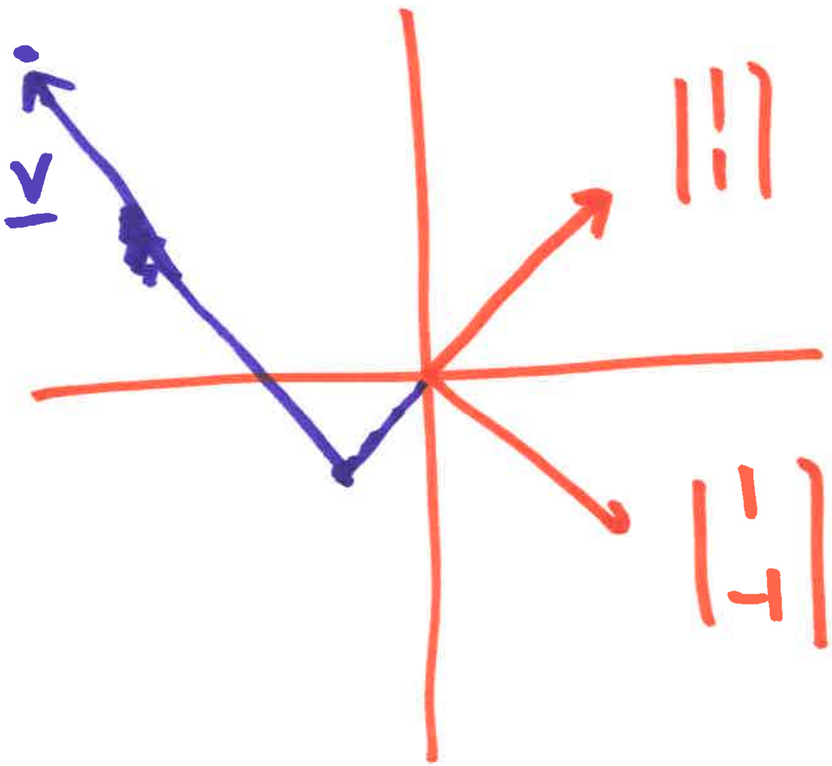
$$[\underline{v}]_B = P_B^{-1}(\underline{v})$$

P_B represented by matrix

$$P_B = \begin{bmatrix} | & | \\ \underline{b}_1 & \underline{b}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$P_B^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\underline{v}]_B = P_B^{-1}(\underline{v}) = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$



$$= -\frac{1}{2}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}$$

More democratic notation:

$P_{\mathcal{B}}^{-1} \rightsquigarrow \underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ Takes $\mathcal{E}$-coords to $\mathcal{B}$-coords

$P_{\mathcal{B}} \rightsquigarrow \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ Takes $\mathcal{B}$-coords to $\mathcal{E}$-coords

<u>Note</u>: $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ and $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ are inverses

Exer Find basis $B = \{\underline{b}_1, \underline{b}_2\}$ of $\mathbb{R}^2$

so that $\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

In other words

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1 \cdot \underline{b}_1 + 2\underline{b}_2, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \cdot \underline{b}_1 + (-1) \cdot \underline{b}_2$$

Alteratively

$$\underset{B \leftarrow \mathcal{E}}{P}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \underset{B \leftarrow \mathcal{E}}{P}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Soln We know $\underset{B \leftarrow \mathcal{E}}{P} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

$$\underset{\mathcal{E} \leftarrow B}{P} = \underset{B \leftarrow \mathcal{E}}{P}^{-1} = -1 \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Recall $P_B = \underset{\mathcal{E} \leftarrow B}{P} = \begin{bmatrix} | & | \\ \underline{b}_1 & \underline{b}_2 \\ | & | \end{bmatrix}$

So $\underline{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underline{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

What if neither basis is std basis?

<u>Exer</u> Suppose $\underline{v}$ in $\mathbb{R}^2$ has coords

$$[\underline{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ wrt } B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Find $[\underline{v}]_C$ wrt $C = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

Soln Pass through std basis as intermediate step.

$$\underset{C \leftarrow B}{P} = \left(\underset{C \leftarrow \mathcal{E}}{P}\right)\left(\underset{\mathcal{E} \leftarrow B}{P}\right)$$

$$= P_C^{-1} P_B$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

<u>Conclusion</u> $[\underline{v}]_C = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

__Theorem__ If $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$, $\mathcal{C} = \{\underline{c}_1, \ldots, \underline{c}_n\}$ are bases for $\mathbb{R}^n$

Then $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[\underline{b}_1]_{\mathcal{C}} \cdots [\underline{b}_n]_{\mathcal{C}} \right]$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[[\underline{c}_1]_{\mathcal{B}} \cdots [\underline{c}_n]_{\mathcal{B}} \right]$$

__Rmks__ 1) $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ and $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ are inverses.

2) For $\mathcal{C} = \mathcal{E}$, we recover $P_{\mathcal{B}} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \left[\underline{b}_1 \cdots \underline{b}_n \right]$