Good morning and welcome to

<u>Lecture 11</u> Bases, dim, rank!

<u>Today</u>: extra office hours 2-3:30pm
9 <u>Lewis</u>

<u>Friday</u>: Quiz through §4.4

<u>Tuesday</u> <u>10/6</u>: Midterm 1 during
lecture meeting

Warmup 1) Which of the following are isomorphisms?

i) $T: P_2 \to \mathbb{R}^3 \qquad T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$

ii) $T: \mathbb{R}^4 \to P_2 \qquad T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a+b) + (b+c)x + (c+d)x^2$

iii) $T: M_{2\times 2} \to P_3 \qquad T(A) = a + bx + cx^2 + dx^3$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

iv) $T: S \to S \qquad T(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots)$

<u>Soln</u> i) $T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix} = \begin{bmatrix} a_0 \\ a_0 + a_1 + a_2 \\ a_0 + 2a_1 + 4a_2 \end{bmatrix}$

$$p(x) = a_0 + a_1 x + a_2 x^2$$

<u>Isom</u> since $Null(T) = \langle 0 \rangle$

$Image(T) = \mathbb{R}^3$

so <u>inj</u> and <u>surj</u>.

ii) <u>Not isom</u> since $Null(T) \neq \langle 0 \rangle$

so <u>not</u> <u>inj</u>

for example $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \in Null(T)$

iii) Isom $T^{-1} : P_3 \to M_{2\times 2}$

$$T^{-1}(p) = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

iv) Not isom   not surj

for example $(1, 0, 0, \ldots) \notin \mathrm{Image}(T)$

Warmup 2) Find isom. $P : \mathbb{R}^2 \longrightarrow V$

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + 2x_2 + 6x_3 = 0 \right\}$$

Soln Find basis B of V.

V = solns to $A\underline{x} = \underline{0}$ where $A = [1 \quad 2 \quad 6]$

Basis B: $\underline{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$

Take $P\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = P_B\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = a_1\underline{v}_1 + a_2\underline{v}_2$

($P_B^{-1}(\underline{v}) = [\underline{v}]_B$ coords of $\underline{v}$ w.r.t. B)

<u>Summary</u> of why we <u>love</u> bases

1) V vect sp, B basis of V $= \{\underline{b}_1, \dots, \underline{b}_n\}$

Then we have an isom.

$$P_B : \mathbb{R}^n \longrightarrow V$$

$$P_B\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 \underline{b}_1 + \dots + a_n \underline{b}_n$$

with inverse

$$P_B^{-1} : V \longrightarrow \mathbb{R}^n$$

$$P_B^{-1}(\underline{v}) = [\underline{v}]_B \quad \text{coords of } \underline{v} \text{ w.r.t. } B$$

2) V, W vect sps, $B = \{\underline{b}_1, \dots, \underline{b}_n\}$, $C = \{\underline{c}_1, \dots, \underline{c}_m\}$ bases of V, W respectively

Then any lin transf $T: V \to W$ can be expressed as an $m \times n$ matrix

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle P_B}\uparrow & & \downarrow{\scriptstyle P_C^{-1}} \\ \mathbb{R}^n & \xrightarrow{P_C^{-1} T P_B} & \mathbb{R}^m \end{array}$$

matrix of $P_C^{-1} T P_B$

$$A = \underbrace{\begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}}_{n} \Big\} m$$

where cols of A are given by

$$\underline{u}_i = \left[T(\underline{b}_i) \right]_C$$

$$\underline{e}_i \xmapsto{P_B} \underline{b}_i \xmapsto{\quad T \quad} T(\underline{b}_i) \xrightarrow{P_C^{-1}} \left[T(\underline{b}_i) \right]_C$$

Exer 1) Find isom $P: \mathbb{R}^4 \to P_3$ by choosing basis B of P_3

2) Find matrix of $T: P_3 \to P_3$ wrt basis B

$$T(p) = x\frac{dp}{dx}$$

Soln 1) Take $B = \{1, x, x^2, x^3\}$

$$P\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = P_B\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = a \cdot 1 + b \cdot x + c \cdot x^2 + d \cdot x^3$$

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Apply with bases $B = C$.

$$T(1) = 0, \quad T(x) = x, \quad T(x^2) = 2x^2, \quad T(x^3) = 3x^3$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

<u>Exer</u> Let $T: V \to W$ be an <u>inj</u> lin transf where $V = \mathbb{R}^2$, $W = \mathbb{R}^3$

Show there are respective bases B, C so that matrix of T takes the form

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

<u>Soln</u> Take basis $\mathcal{B}$ to be <u>any</u> basis of $\mathbb{R}^2$, for example take standard basis $\underline{b}_1 = \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{b}_2 = \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Let $\underline{w}_1 = T(\underline{b}_1)$, $\underline{w}_2 = T(\underline{b}_2)$

<u>Observe</u>: $\underline{w}_1, \underline{w}_2$ are lin indep. since T is <u>inj</u> (check this!)

Choose $\underline{w}_3$ to be any vector <u>not</u> in $\text{span}\{\underline{w}_1, \underline{w}_2\}$

<u>Observe</u> $\underline{w}_1, \underline{w}_2, \underline{w}_3$ is a basis

take this to be C (check this!)

Now the <u>fun part</u> calculate matrix of T wrt B, C

$$A = \begin{bmatrix} | & | \\ [T(\underline{b}_1)]_C & [T(\underline{b}_2)]_C \\ | & | \end{bmatrix}$$

$$T(\underline{b}_1) = \underline{w}_1 = \underline{1} \cdot \underline{w}_1 + \underline{0} \cdot \underline{w}_2 + \underline{0} \cdot \underline{w}_3$$

$$T(\underline{b}_2) = \underline{w}_2 = \underline{0} \cdot \underline{w}_1 + \underline{1} \cdot \underline{w}_2 + \underline{0} \cdot \underline{w}_3$$

Conclusion: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Wow!

Further applications of bases:

Thm: V vect sp with basis $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$

Then 1) $\underline{v}_1, \ldots, \underline{v}_k$ lin indep $\Rightarrow k \leq n$

2) $\underline{v}_1, \ldots, \underline{v}_k$ span $\Rightarrow k \geq n$

Rmk We already have seen this for $V = \mathbb{R}^n$.

<u>Proof of</u> 1) (Leave 2) as an exer)

Suppose $\underline{v}_1, \ldots, \underline{v}_k$ lin indep but $k > n$.

Apply $P_B^{-1} : V \longrightarrow \mathbb{R}^n$ isom.

$P_B^{-1}(\underline{v}_1), \ldots, P_B^{-1}(\underline{v}_k)$ must be lin dep since $k > n$

So there are $c_1, \ldots, c_k$ <u>not</u> all 0

so that $c_1 P_B^{-1}(\underline{v}_1) + \cdots + c_k P_B^{-1}(\underline{v}_k) = \underline{0}$

Apply $P_B : \mathbb{R}^n \to V$ inverse of P_B^{-1}

$$P_B(c_1 P_B^{-1}(\underline{v}_1) + \dots + c_k P_B^{-1}(\underline{v}_k)) = P_B(\underline{0})$$

$$c_1 \underline{v}_1 + \dots + c_k v_k = \underline{0}$$

So $\underline{v}_1, \dots, \underline{v}_k$ are lin dep. $\lightning$

Contradiction so must have $k \le n$.

Def V vect sp

$\dim V$ = size of any basis
if finite basis exists

else $\dim V = \infty$

Examples: 1) $\dim \mathbb{R}^n = n$

2) $\dim P_n = n+1$

3) $\dim S = \infty$

Thm (dim is well-defined)

V vect sp with bases $B = \{\underline{b}_1, \dots, \underline{b}_n\}$
$C = \{\underline{c}_1, \dots, \underline{c}_k\}$

then $n = k$

Proof $\underline{c}_1, \dots, \underline{c}_k$ basis so $\underline{c}_1, \dots, \underline{c}_k$ lin indep. in vect sp V with basis $\underset{=}{B}$ of size n
$\{\underline{b}_1, \dots, \underline{b}_n\}$

Previous Thm tells us ~~[illegible]~~ $k \leq n$

Similarly can see $n \leq k$.

So $n = k$!