Lecture 10 Focus on Linear Transformations

Today: usual Office Hours, 1-3pm, 736 Evans

Thursday: extra Office Hours, 2-3:30pm
9 Lewis

Friday: Quiz through §4.4

Tuesday 10/6 Midterm 1 during lecture time
material through §4.4

Warmup $V = P_2 = \{ f : \mathbb{R} \to \mathbb{R}$ poly fn of deg $\leq 2 \}$

Find coords of $f = 2 + 3x + x^2$ with respect to bases:

1) std basis $B = \{1, x, x^2\}$

2) alternative basis $B = \{x, 1, 1 - x^2\}$

Soln 1) $f = 2 + 3x + x^2 = 2 \cdot 1 + 3 \cdot x + 1 \cdot x^2$

$$[f]_B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

2) Want to solve $f = 2 + 3x + x^2 =$

$a_1 \cdot x + a_2 \cdot 1 + a_3 \cdot (1 - x^2)$

Convert into augmented matrix

$$\left[\begin{array}{ccc:c} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

<u>Soln</u>: $a_1 = 3$
$a_2 = 3$
$a_3 = -1$

$$[f]_B = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$$

Reminder A $m \times n$ matrix

matrix mult. gives lin transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(\underline{v}) = A\underline{v}$$

Properties of lin transf:

1) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$
2) $T(c\underline{u}) = c\,T(\underline{u})$

Amazing fact: any lin transf. is represented by a matrix

$$A = \begin{bmatrix} | & & | \\ T(\underline{e}_1) & \cdots & T(\underline{e}_n) \\ | & & | \end{bmatrix}$$

Today: repeat everything we know now for lin transf $T: V \to W$

vector spaces

means transf satisfying two properties recalled above.

Key concepts

1) Image/Range of $T: V \to W$

$$\text{Image}\, T = \{ \underline{w} \in W \mid \text{there is } \underline{v} \in V \text{ so that } \underline{w} = T\underline{v} \}$$

2) Null space/Kernel of $T: V \to W$

$$\text{Null}\, T = \{ \underline{v} \in V \mid T(\underline{v}) = \underline{0} \}$$

3) Solving eqn $T\underline{x} = \underline{b}$ for $\underline{x} \in V$, $\underline{b} \in W$

<u>Familiar facts:</u>

1) Image T is subspace of W

   T is <u>surj</u> if Image $T = W$

2) Null T is subspace of V

   T is <u>inj</u> if Null $T = \{0\}$

Exer 1) $V = W = P_3$, $T : V \to W$

$$T(f) = \frac{df}{dx}$$

Find bases for Image (T), Null (T)

Soln If $f = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

then $T(f) = a_1 + 2a_2 x + 3a_3 x^2$

Basis for Image T $= \{1, x, x^2\}$

Basis for Null $T = \{1\}$

(Observation for later:

$$4 = 3 + 1$$

)

<u>Exer2</u>) Let $V = P_2$, $W = \mathbb{R}^2$

$T : V \to W$ for $p = a_0 + a_1x + a_2x^2$

$$T(p) = \begin{bmatrix} p(1) \\ \frac{dp}{dx}(0) \end{bmatrix}$$

(<u>Exer</u>: check this is a lin transf)

Find soln set of

$$T(p) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Soln $T(p) = \begin{bmatrix} p(1) \\ \frac{dp}{dx}(0) \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + a_2 \\ a_1 \end{bmatrix}$

$T(p) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ means $\begin{matrix} a_0 + a_1 + a_2 = -1 \\ a_1 = 1 \end{matrix}$

Soln set: $\left\{ \begin{matrix} p = a_0 + 1 \cdot x + a_2 x^2 \\ a_0 + a_2 = -2 \end{matrix} \right\}$

Exer 3) Let $V = W = M_{2\times 2}$

$$T : M_{2\times 2} \longrightarrow M_{2\times 2}$$

$$T(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot A$$

(Exer: check this is a lin transf)

Find bases for

Null T , Image T

$\underline{\text{Soln}}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(A) = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

$\text{Null}\, T = \left\{ A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right\}$

Basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

$\text{Image}\, T = \left\{ B = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \right\}$

Basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

<u>Note:</u>

$4 = 2 + 2$

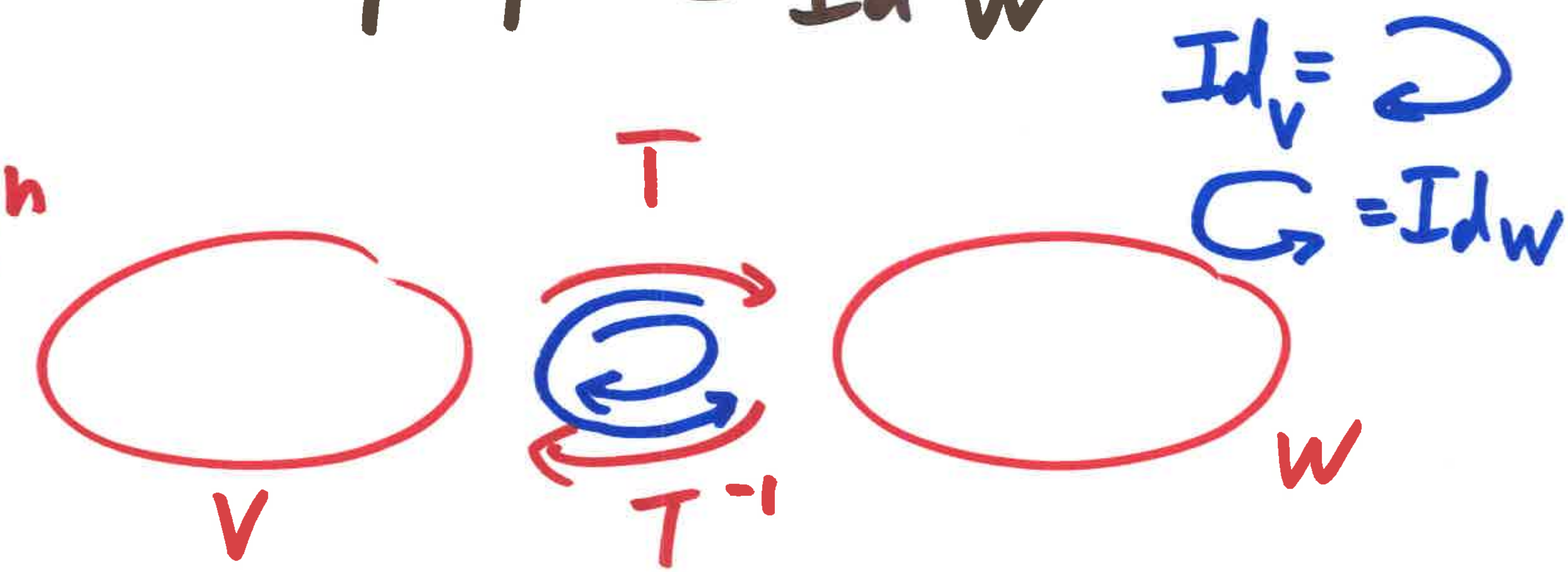
<u>Dirty secret</u>: If $T: V \to W$ is lin transf, and we choose bases of V, W, then we can express T as a matrix!

We'll explain this now.

__Def.__ A lin transf $T: V \to W$ is called an __isomorphism__ if there is an inverse $T^{-1}: W \to V$ with

$$T^{-1}T = Id_V$$
$$TT^{-1} = Id_W$$

__Cartoon__



<u>Theorem</u>: Suppose vect sp V has basis

$$B = \{ \underline{b}_1, \dots, \underline{b}_n \}$$

Then we have an isomorphism

$$P_B : \mathbb{R}^n \longrightarrow V$$

$$P_B \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 \underline{b}_1 + \dots + a_n \underline{b}_n$$

<u>Why true</u>? Inverse

$$P_B^{-1} : V \longrightarrow \mathbb{R}^n$$

is given by taking coords

$$P_B^{-1}(\underline{v}) = [\underline{v}]_B$$

<u>Exer</u> Check this works.

Now let's associate to any lin transf $T : V \to W$ a matrix

Suppose $B = \{\underline{b}_1, \dots, \underline{b}_n\}$, $C = \{\underline{c}_1, \dots, \underline{c}_m\}$ are bases of V, W respectively

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ {\scriptstyle P_B}\big\uparrow & & \big\downarrow{\scriptstyle P_C^{-1}} \\ \mathbb{R}^n & \xrightarrow{\; P_C^{-1} T P_B \;} & \mathbb{R}^m \end{array}$$

Let's unpack this construction

$$P_C^{-1} T P_B(\underline{v}) = P_C^{-1} T(a_1 \underline{b}_1 + \cdots + a_n \underline{b}_n)$$

$$\underline{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad = P_C^{-1}(a_1 T(\underline{b}_1) + \cdots + a_n T(\underline{b}_n))$$

$$= \left[a_1 T(\underline{b}_1) + \cdots + a_n T(\underline{b}_n)\right]_C$$

Resulting matrix

$$\underset{\sim}{A} = \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}$$

where $\underline{u}_i = \left[T(\underline{b}_i) \right]_C$

Exer Find matrix of lin transf

$$T : P_3 \to P_2 \quad T(p) = x \frac{d^2 p}{dx^2}$$

with respect to bases

$$B = \{1, x, x^2, x^3\}$$

$$C = \{1, x, x^2\}$$

<u>Soln</u> $A = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 & \underline{u}_4 \end{bmatrix}$ $\quad \underline{u}_i = [T(\underline{e}_i)]_C$

$\underline{e}_1 = 1, \ \underline{e}_2 = x, \ \underline{e}_3 = x^2, \ \underline{e}_4 = x^3$

$T(\underline{e}_1) = 0, \ T(\underline{e}_2) = 0, \ T(\underline{e}_3) = 2x, \ T(\underline{e}_4) = 6x^2$

$= 0 \cdot 1$	$= 0 \cdot 1$	$= 0 \cdot 1$	$= 0 \cdot 1$
$+ 0 \cdot x$	$+ 0 \cdot x$	$+ 2 \cdot x$	$+ 0 \cdot x$
$+ 0 \cdot x^2$	$+ 0 \cdot x^2$	$+ 0 \cdot x^2$	$+ 6 \cdot x^2$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

A lot less scary than

$$T(p) = x \frac{d^2 p}{dx^2} \quad !$$