Welcome to Math 54 Review

Part I !

Orthogonality for the moment

$V = \mathbb{R}^n$, $\langle \underline{v}, \underline{w} \rangle = \underline{v} \cdot \underline{w}$

$= v_1 w_1 + \cdots + v_n w_n$

$\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

__Example__ $A = m\left\{\underbrace{\begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix}}_{n}\right.$ $m \times n$ matrix

1) What is $\mathrm{Row}(A)^{\perp}$?

$$A = \begin{bmatrix} — & \underline{a}_1 & — \\ — & \underline{a}_2 & — \\ & \vdots & \\ — & \underline{a}_m & — \end{bmatrix}$$

$$\mathrm{Row}(A) = \mathrm{Span}\{\underline{a}_1, \dots, \underline{a}_m\} \subset \mathbb{R}^n$$

__Answer__ $\mathrm{Row}(A)^{\perp} = \mathrm{Null}(A)$

2) What is $\mathrm{Col}(A)^\perp$

$$A = \begin{bmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_n \\ | & & | \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{Span}\{\underline{b}_1, \ldots, \underline{b}_n\} \subset \mathbb{R}^m$$

<u>Answer</u> $\mathrm{Col}(A)^\perp = \mathrm{Row}(A^T)^\perp$

$$= \mathrm{Null}(A^T)$$

Gram-Schmidt:

$$\boxed{\begin{array}{l}\text{lin indep}\\ \underline{v}_1,\dots,\underline{v}_k\end{array}} \xrightarrow{\text{G-S}} \boxed{\begin{array}{l}\text{orthog set}\\ \underline{u}_1,\dots,\underline{u}_k\end{array}}$$

key prop: $\text{Span}\{\underline{v}_1,\dots,\underline{v}_k\} = \text{Span}\{\underline{u}_1,\dots,\underline{u}_k\}$

in fact: $\text{Span}\{\underline{v}_1,\dots,\underline{v}_\ell\} = \text{Span}\{\underline{u}_1,\dots,\underline{u}_\ell\}$

$\ell \le k$

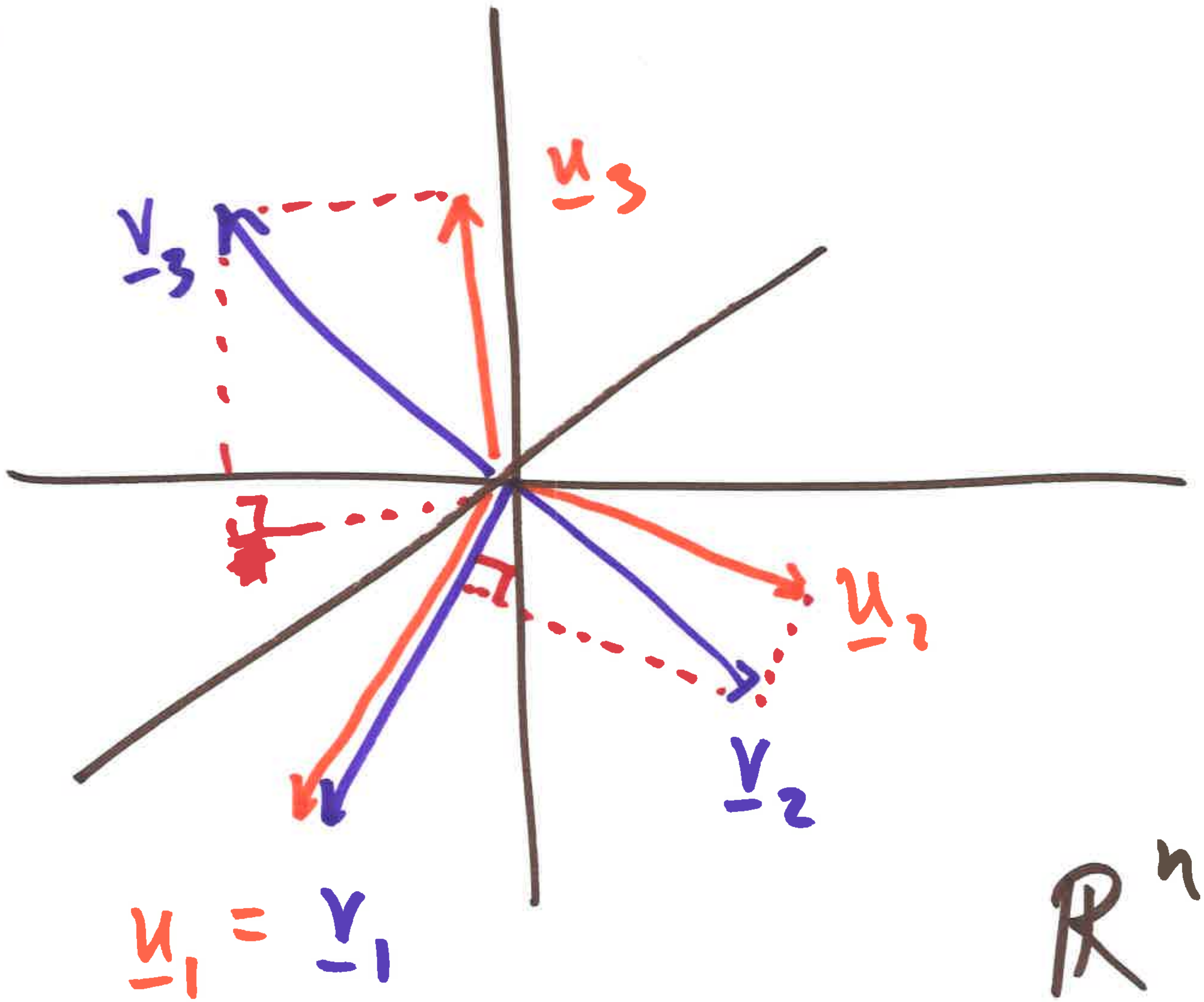
Formulas

$$\underline{u}_1 = \underline{v}_1$$

$$\underline{u}_2 = \underline{v}_2 - \frac{\langle \underline{v}_2, \underline{u}_1 \rangle}{\langle \underline{u}_1, \underline{u}_1 \rangle} \cdot \underline{u}_1$$

$$\underline{u}_3 = \underline{v}_3 - \frac{\langle \underline{v}_3, \underline{u}_1 \rangle}{\langle \underline{u}_1, \underline{u}_1 \rangle} \underline{u}_1 - \frac{\langle \underline{v}_3, \underline{u}_2 \rangle}{\langle \underline{u}_2, \underline{u}_2 \rangle} \underline{u}_2$$

$$\vdots$$



$\underline{v}_3$
$\underline{u}_3$
$\underline{u}_2$
$\underline{v}_2$
$\underline{u}_1 = \underline{v}_1$
$\mathbb{R}^n$

Rewrite formulas:

$$\underline{v}_1 = \underline{u}_1$$

$$\underline{v}_2 = \underline{u}_2 + \frac{\langle \underline{v}_2, \underline{u}_1 \rangle}{\langle \underline{u}_1, \underline{u}_1 \rangle} \underline{u}_1$$

$$\underline{v}_3 = \underline{u}_3 + \frac{\langle \underline{v}_3, \underline{u}_1 \rangle}{\langle \underline{u}_1, \underline{u}_1 \rangle} \underline{u}_1 + \frac{\langle \underline{v}_3, \underline{u}_2 \rangle}{\langle \underline{u}_2, \underline{u}_2 \rangle} \underline{u}_2$$

$A = QR$ factorization

$$A = \left.\underbrace{\begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_k \\ | & & | \end{bmatrix}}_{k}\right\} n \qquad n \times k \text{ matrix}$$

$$Q = \begin{bmatrix} | & & | \\ \hat{\underline{u}}_1 & \cdots & \hat{\underline{u}}_k \\ | & & | \end{bmatrix} \qquad n \times k \text{ matrix}$$

with ~~on~~ orthonormal cols.

$$\hat{\underline{u}}_i = \frac{1}{\|\underline{u}_i\|} \cdot \underline{u}_i$$

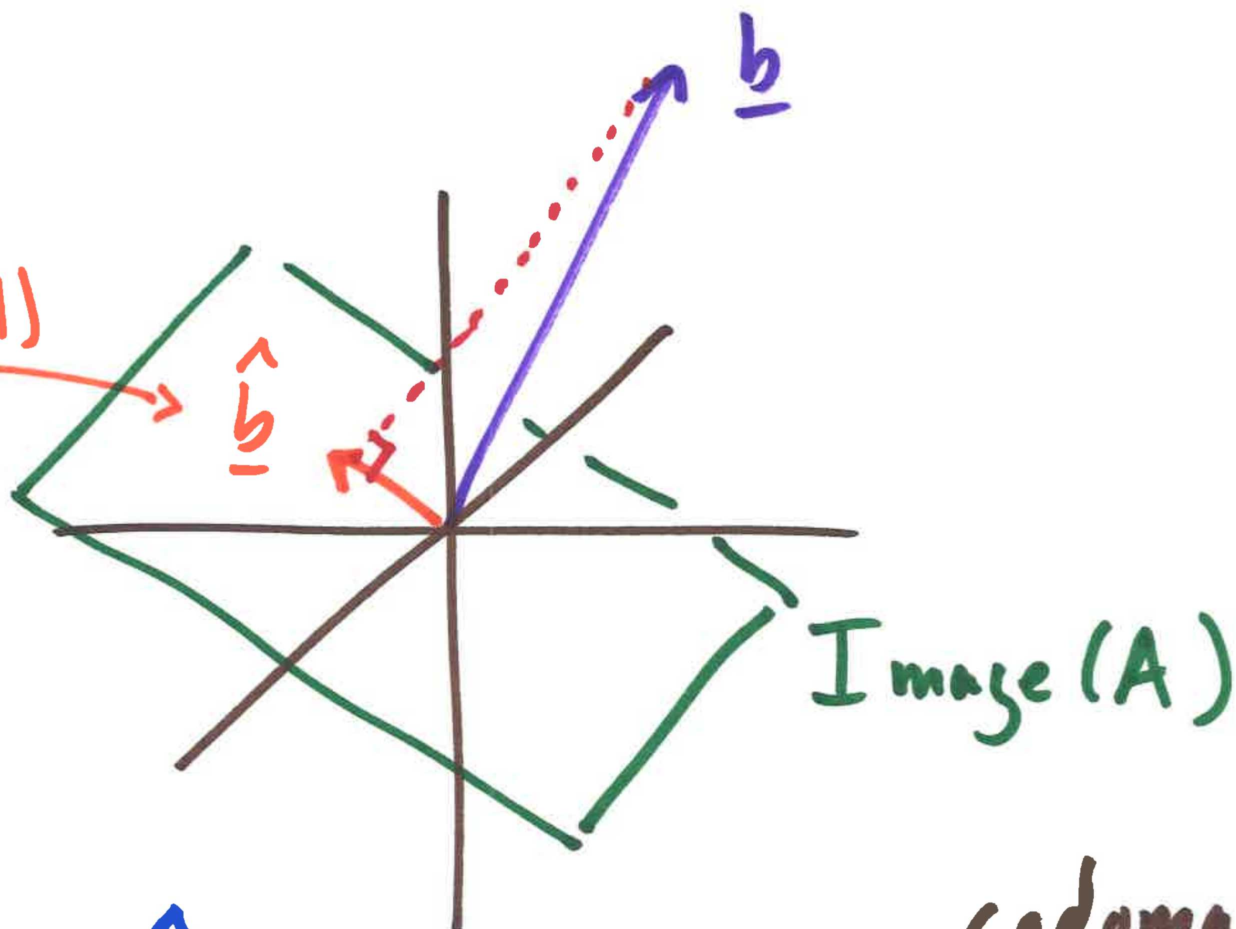
Formula expressing $\underline{v}_1, \ldots, \underline{v}_k$ in terms of $\underline{u}_1, \ldots, \underline{u}_k$ gives

$$A = Q \underbrace{R}_{k \times k \text{ upper } \Delta\text{ar matrix}}$$

<u>Least</u> squares approx $A\underline{x} = \underline{b}$

<u>Cartoon</u>

$\hat{\underline{b}} \in \text{Image}(A)$
closest
to $\underline{b}$



<u>Solve</u> $A\hat{\underline{x}} = \hat{\underline{b}}$

codomain
$\mathbb{R}^m$

Formula for $\hat{\underline{x}}$

Solving $A\hat{\underline{x}} = \hat{\underline{b}}$ is same as

solving $\underbrace{A^TA}_{B}\underline{x} = \underbrace{A^T\underline{b}}_{\underline{y}}$

$B\underline{x} = \underline{y}$

Inner Product Space V = vect sp

$\langle\ ,\ \rangle$ = inner product

satisfying some axioms...

Ex: 1) $V = \mathbb{R}^n$, $\langle\ ,\ \rangle$ = dot product

[**Note** this is not only inner prod on $\mathbb{R}^n$

for ex, $V = \mathbb{R}^n$

$$\langle \underline{v}, \underline{w} \rangle = \underline{v}^T A \underline{w}$$

where A is $n \times n$ sym. matrix with pos e-values]

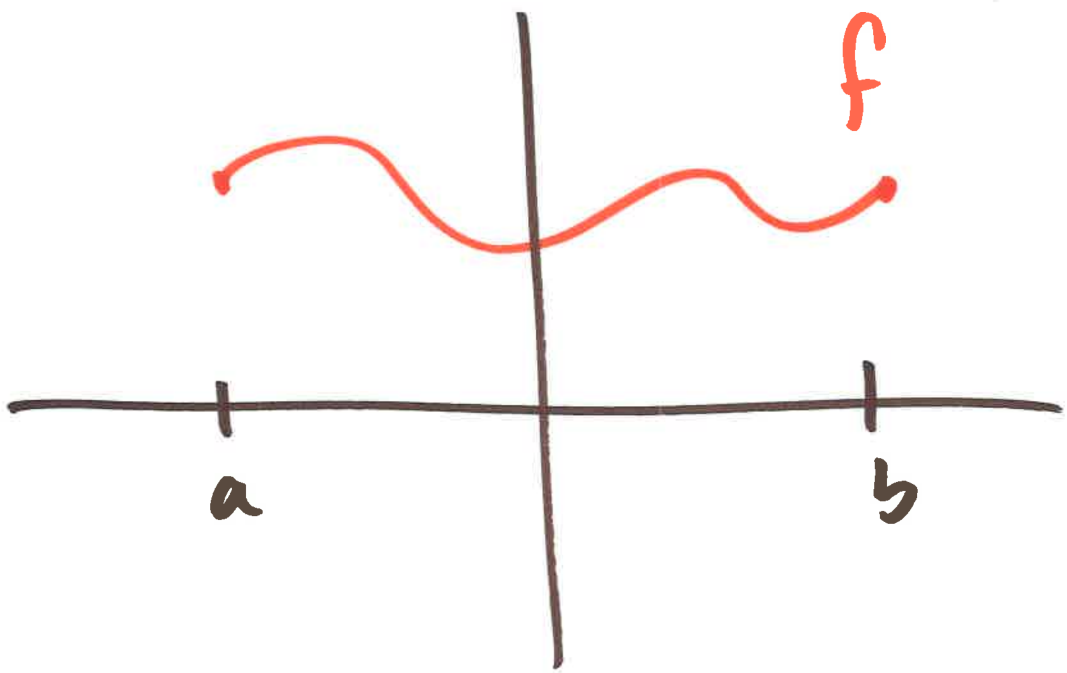
2) $V = P_n = \{\text{polys of deg} \leq n.\}$

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + \cdots + p(n)q(n)$$

3) $V = \{ f : [a,b] \longrightarrow \mathbb{R} \}$

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$



Repeat all geometry from $\mathbb{R}^n$ with dot prod to abstract inner prod space V with inner prod $\langle \, , \, \rangle$

<u>Ex</u> Gram-Schmidt in function sp such as

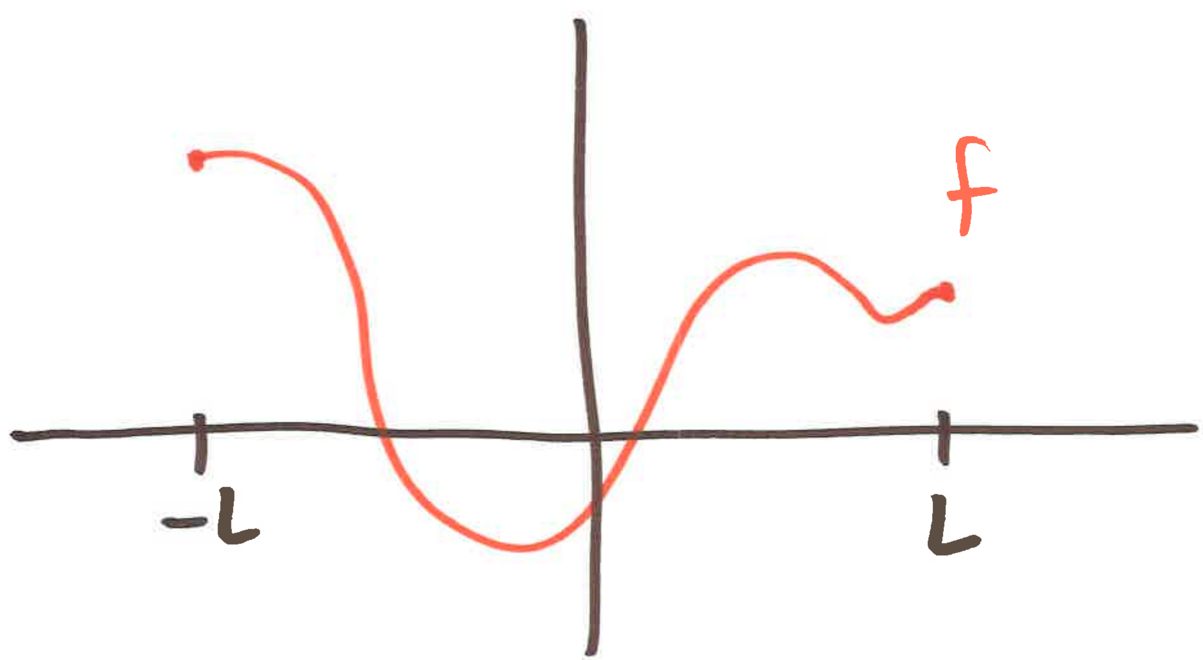
$$V = P_n = \{\text{polys of deg} \le n\}$$

$$V = \{ f : [a,b] \to \mathbb{R} \}$$

Focus on inner prod sp

$$V = \{ f : [-L, L] \longrightarrow \mathbb{R} \}$$

$$\langle f, g \rangle = \int_{-L}^{L} f(x) g(x) \, dx$$

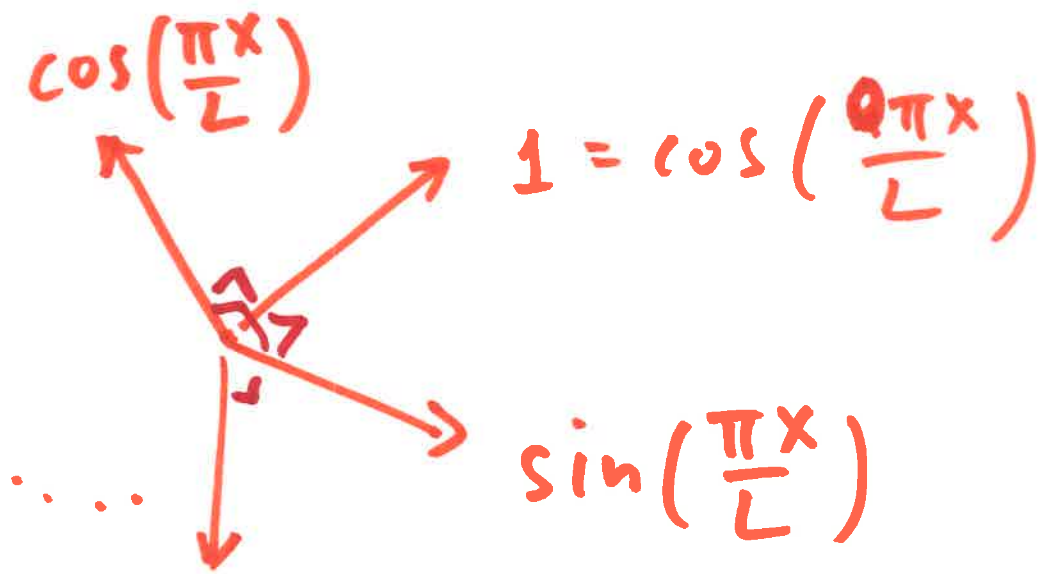


Beautiful collection of <u>orthog</u> <u>vectors</u>

$$\cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, \ldots$$

$$\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

<u>Cartoon</u>

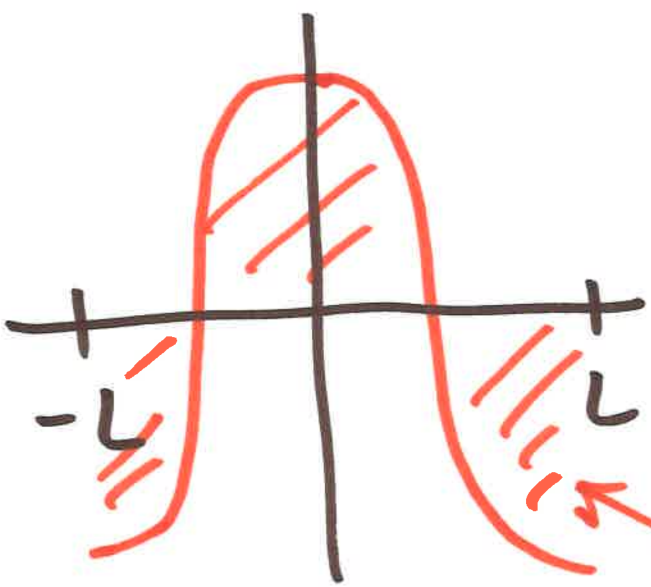


Ex of orthogonality:

$$\langle \underset{\cos\left(\frac{0\pi x}{L}\right)}{\underset{\parallel}{1}},\ \cos\left(\frac{\pi x}{L}\right) \rangle = \int_{-L}^{L} 1\cdot\cos\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)\Big|_{-L}^{L}$$

$$= 0 - 0 = 0$$



Area = 0

<u>Note</u> not orthonormal

$$\langle 1, 1 \rangle = 2L$$

$$\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle = L$$

$$n > 0$$

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = L$$