Name (Last, First): ______________________________

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(1) Let A be a 6×8 matrix. Suppose that the null space of A has a basis consisting of 3 vectors. Do the columns of A span $\mathbb{R}^6$? Justify your answer.

Solution: According to the Rank Theorem (Section 2.7, Theorem 13), we have $\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = 8$. We are given that $\dim \operatorname{Nul}(A) = 3$. Thus $\operatorname{rank}(A) = 5$. By Section 2.6 Theorem 12, the pivot columns of A form a basis for the column space of A, so there are 5 pivot columns. Thus there are 5 pivot rows. This means there is a row which does not contain a pivot; thus the columns of A do not span $\mathbb{R}^6$.

(2) Compute the determinant of the following matrix.

$$\begin{bmatrix} 3 & -1 & 0 & 2 \\ 4 & 9 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ -1 & -2 & 0 & 2 \end{bmatrix}$$

Solution: We use Section 3.1, Theorem 1 to compute the determinant:

$$\begin{aligned}
\det \begin{bmatrix} 3 & -1 & 0 & 2 \\ 4 & 9 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ -1 & -2 & 0 & 2 \end{bmatrix} &\overset{(1)}{=} 0 \cdot (-1)^{1+3} \det \begin{bmatrix} 4 & 9 & 0 \\ 0 & 1 & 3 \\ -1 & -2 & 2 \end{bmatrix} + 2 \cdot (-1)^{2+3} \det \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 3 \\ -1 & -2 & 2 \end{bmatrix} \\
&\quad + 0 \cdot (-1)^{3+3} \det \begin{bmatrix} 3 & -1 & 2 \\ 4 & 9 & 0 \\ -1 & -2 & 2 \end{bmatrix} + 0 \cdot (-1)^{4+3} \det \begin{bmatrix} 3 & -1 & 2 \\ 4 & 9 & 0 \\ 0 & 1 & 3 \end{bmatrix} \\
&= 2 \cdot (-1)^{2+3} \det \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 3 \\ -1 & -2 & 2 \end{bmatrix} \\
&\overset{(2)}{=} 2 \cdot (-1)^{2+3} \Bigg(3 \cdot (-1)^{1+1} \det \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} + 0 \cdot (-1)^{2+1} \det \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix} \\
&\quad + (-1) \cdot (-1)^{3+1} \det \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \Bigg) \\
&= 2 \cdot (-1)^{2+3} \Big(3 \cdot (-1)^{1+1} (8) + (-1) \cdot (-1)^{3+1} (-5) \Big) \\
&= -58\,.
\end{aligned}$$

In the step marked (1), we use the cofactor expansion across the 3rd column; in the step marked (2), we use the cofactor expansion across the 1st column.