Lecture 22 Everything is Illuminated and First Order!

No Office Hours today

Friday: Quiz through 4.5

Warmup 0 Show $1, e^{t^2}$ can not be basis of solns for linear 2^{nd} order eqn

Soln We'll contradict the fact that if they were a basis, we could solve IVP for any given values

Consider Wronskian

$$W(t) = \det \begin{bmatrix} 1 & e^{t^2} \\ 0 & 2te^{t^2} \end{bmatrix} = 2te^{t^2}$$

At $t=0$, $W(0) = 0$

since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are not a basis of $\mathbb{R}^2$

1st col at $t=0$ 2nd col at $t=0$

So can't solve IVP with

$y(0) = 0$

$y'(0) = 1$

Warmup! Write the following 3rd order IVP as a 1st order IVP

$$y''' + 7y' - 3y = \sin(t)$$

$$y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 5$$

Soln Introduce new fns

$$y_1 = y \qquad y_2 = y' \qquad y_3 = y''$$

These fns satisfy

$$y_1' = y_2$$

$$y_2' = y_3$$

$$y_3' = -7y_2 + 3y_1 + \sin(t)$$

$$y_1(0) = 1, \quad y_2(0) = -2, \quad y_3(0) = 5$$

Can rewrite as matrix eqn

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \underline{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} \qquad \underline{y}_0 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

$$\underline{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -7 & 0 \end{bmatrix} \underline{y} + \begin{bmatrix} 0 \\ 0 \\ \sin(t) \end{bmatrix}$$

~~yua~~ $\underline{y}(0) = \underline{y}_0$

Def. A normal form for a lin syst of diff eqns is

$$\underline{y}' = A\,\underline{y} + \underline{f}$$

vector of fns

matrix of fns

Warmup 2 Rewrite the following 2^{nd} order system as a 1^{st} order system

$$x_1'' + 6\sin(t)\,x_1 - 2x_2' = 0$$

$$x_2'' - e^{3t}x_1 = 0$$

Soln: Introduce new fns

$$y_1 = x_1, \quad y_2 = x_1', \quad y_3 = x_2, \quad y_4 = x_2'$$

These satisfy eqns

$$y_1' = y_2$$

$$y_3' = y_4$$

$$y_2' = -6\sin(t)\, y_1 + 2y_4$$

$$y_4' = e^{3t}\, y_1 \qquad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\underline{y}' = A\,\underline{y} \quad \text{where}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6\sin(t) & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ e^{3t} & 0 & 0 & 0 \end{bmatrix}$$

1st Order Lin. Syst. IVP Thm

$A(t)$, $\underline{f}(t)$ cont. on (a,b)

$n \times n$ matrix of fns

n-vector of fns

t_0 pt of (a,b)

Then there exists unique soln to IVP

$$\underline{y}'(t) = A(t)\,\underline{y}(t) + \underline{f}(t)$$

$$\underline{y}(t_0) = \underline{y}_0$$

any n-vector of numbers

Brief review of usual analysis in this new context:

Diff op $L = \begin{bmatrix} \frac{d}{dt} & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dt} \end{bmatrix} - A(t)$

(lin transf)

Nonhomog diff eqn $L(\underline{y}) = \underline{f}$

Homog diff eqn $L(\underline{y}) = \underline{0}$

There exists basis of solns to homog eqn

$\underline{y}_1, \dots, \underline{y}_n$

We can find $\underline{y}_p$ a particular soln to nonhomog eqn

General soln is of form

$$\underline{y} = \underline{y}_p + c_1 \underline{y}_1 + \dots + c_n \underline{y}_n$$

To solve IVP, want $\underline{y}(t_0) = \underline{y}_0$

So need to solve lin syst

$$\begin{bmatrix} | \\ \underline{y}_0 - \underline{y}_p(t_0) \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \underline{y}_1 & \cdots & \underline{y}_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Wronskian

$$W(t) = \det \begin{bmatrix} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{bmatrix}$$

$y_1, \ldots, y_n$ basis of solns to homog eqn

$\Rightarrow W(t) \neq 0$ for any t, so can always solve IVP

Now specialize to <u>constant</u> <u>coefs</u>

$$\underline{y}' = A\,\underline{y} + \underline{f}$$

$n \times n$ matrix of <u>numbers</u>

Focus on homog eqn

$$\underline{y}' = A\,\underline{y}$$

Think of $\underline{y}' = A\underline{y}$ as e-vector/e-value eqn

for "$\lambda = \frac{d}{dt}$"

$$A\,\underline{y} = \frac{d}{dt} \cdot \underline{y}$$

<u>Thm</u> If $\underline{u}$ is an e-vector of A with e-value r, then $e^{rt} \cdot \underline{u}$ is a soln to

$$A\underline{y} = \frac{d}{dt}\underline{y}$$

__Check__: $A \cdot (e^{rt}\underline{u}) = e^{rt} A \underline{u}$

$= r e^{rt} \underline{u} = \frac{d}{dt}(e^{rt}\underline{u})$!

<u>Exer</u>: Find gen soln to lin syst

$\underline{y}' = A\underline{y}$ where

1) $A = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$ $\quad r_1 = 0, \; r_2 = 5$

$\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \underline{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\underline{y} = c_1 e^{0 \cdot t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{5t} \end{bmatrix}$$

2) $A = \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix}$

$$\chi_A(r) \underset{\det}{=} \begin{vmatrix} 2-r & -3 \\ 1 & -2-r \end{vmatrix} = (2-r)(-2-r) + 3$$

$$= -4 + 2r - 2r + r^2 + 3$$

$$= r^2 - 1$$

e-values $r_1 = 1, \quad r_2 = -1$

e-vectors: find null space

$\underline{r_1 = 1}$: $\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$ $\quad \underline{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\underline{r_2 = -1}$ $\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$ $\quad \underline{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General soln:

$$\underline{y} = c_1 e^{1 \cdot t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-1 \cdot t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

__Thm__ If A has basis of e-vectors $\underline{u}_1, \dots, \underline{u}_n$ with corr. e-values $r_1, \dots, r_n$ (for example, A symmetric)

then $A\underline{y} = \underline{y}'$ has basis of solns

$$e^{r_1 t}\underline{u}_1, \dots, e^{r_n t}\underline{u}_n$$