Announcements:

1) My Office Hours: Thursdays 12:30-2:30
736 Evans

2) Midterm 2 scheduling: Thursday 10/30

3) Enrollment questions: Thomas Brown
Jennifer Pinney
9th floor Evans

Welcome Back!

Warmup: Solve (if possible!) lin system:

$$x_1 - x_3 + 2x_4 = 1$$
$$2x_1 + x_3 - x_4 = 0$$

$\underset{\substack{\uparrow \\ m}}{2}$ eqns in $\underset{\substack{\uparrow \\ n}}{4}$ vars

$$\left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & -1 & 0 \end{array}\right]$$

$m \times (n+1)$ matrix

$$\rightsquigarrow \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 3 & -5 & -2 \end{array}\right]$$

REF

$$\rightsquigarrow \left[\begin{array}{cccc:c} 1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1 & -5/3 & -2/3 \end{array}\right]$$

RREF

Are there solns? i.e. is syst. consistent?

No $\iff$ Row of the form

$[0 \; \cdots \; 0 \mid b \neq 0]$ REF

$[0 \; \cdots \; 0 \mid 1]$ RREF

Yes $\iff$ Otherwise.

In our example: Pivot vars: x_1, x_3

Others: free vars x_2, x_4

$$\text{Soln set} = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \begin{array}{l} x_2, x_4 \text{ any numbers} \\ x_1 = \frac{1}{3} - \frac{1}{3}x_4 \\ x_3 = -\frac{2}{3} + \frac{5}{3}x_4 \end{array} \right\}$$

__Theorem__ Given any (augmented) matrix it is possible to find a RREF matrix equivalent to the original matrix by row operations. In fact the RREF is unique.

The utility of this statement is that the proof is an __algorithm__.

Algorithm for RREF

leftmost!

Step 1 Find nonzero col.

Step 2 Find nonzero entry of nonzero col
Exchange rows to move entry to top.

Step 3 Create zeros below pivot entry.

Step 4 Ignore row and col containing pivot entry. Repeat with submatrix below & right.

+

Step 5 Scale rows to make pivots = 1. + Create zeros above pivots

Example

$$\begin{bmatrix} 0 & 2 & 0 & 2 & 2 \\ 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & 10 & -3 & 0 \end{bmatrix}$$

$$\boxed{\rightsquigarrow} \begin{bmatrix} 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 10 & -3 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & -3 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 10 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF.

$$\rightsquigarrow \begin{bmatrix} 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & \boxed{1} & -3/10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{RREF}$$

2

We've been studying matrices with a focus on <u>rows</u>.

Now we'll focus on <u>cols</u>.

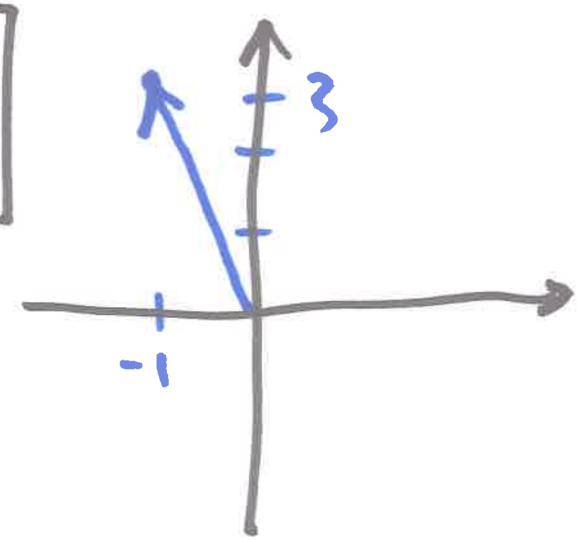
<u>Def</u>. An n-vector / vector / col vector is an ordered list of n numbers.

Draw as an $n \times 1$ matrix:

$$\underline{u} = n\left\{\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{bmatrix}}_{1}\right.$$

<u>Ex</u>: <u>$n=1$</u> $\quad \underline{u} = [-2]$

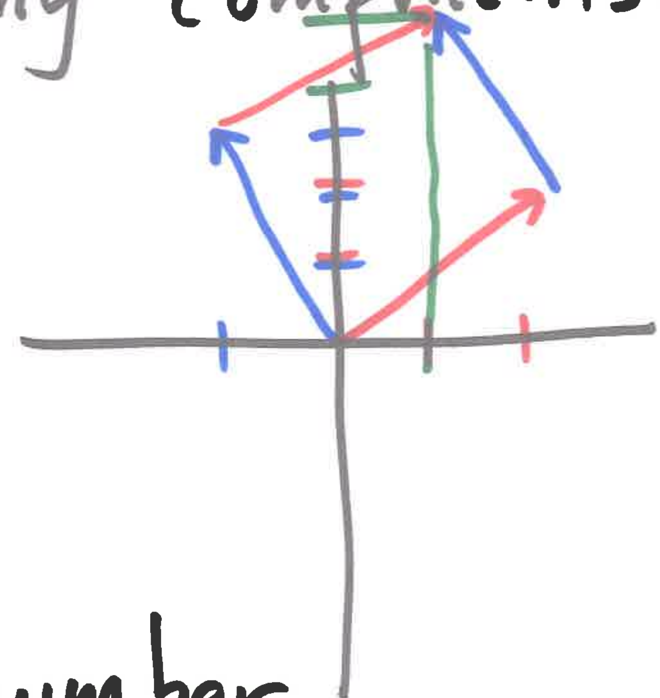
<u>$n=2$</u> $\quad \underline{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$



What can we do with vectors?

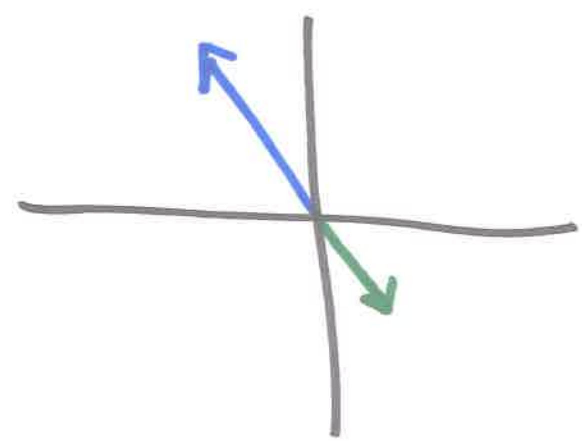
1) Add vectors by adding components:

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



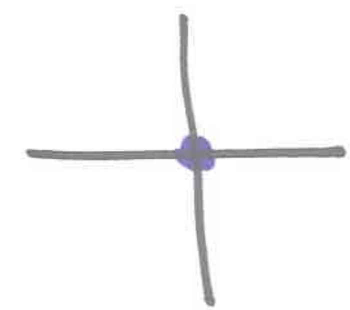
2) Scale vector by number

$$-\frac{1}{2} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$



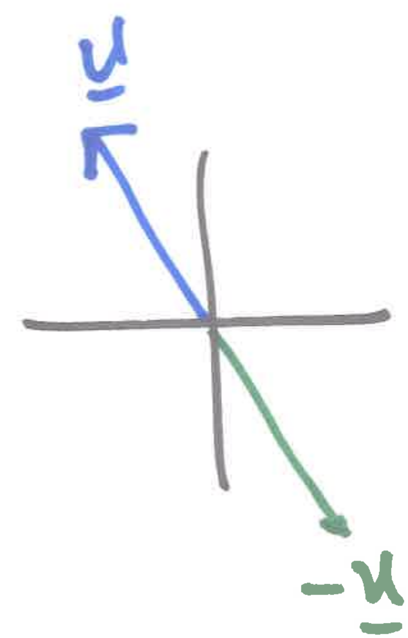
Special vectors:

1) zero vector $\underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$



2) Given vector $\underline{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

opp. vector $-\underline{u} = \begin{bmatrix} -a_1 \\ \vdots \\ -a_n \end{bmatrix}$



Properties

$$\underline{u} + \underline{0} = \underline{u} = \underline{0} + \underline{u}$$

$$\underline{u} + (-\underline{u}) = \underline{0} = (-\underline{u}) + \underline{u}$$

Def A vector $\underline{u}$ is a **lin. combination** of vectors $\underline{v}_1, \dots, \underline{v}_k$ with coeffs. $a_1, \dots, a_k$ if

numbers!

$$\underline{u} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_k \underline{v}_k$$

vectors!

Exer For what c is $\underline{u}$ a lin comb of $\underline{v}_1, \underline{v}_2$?

$$\underline{u} = \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix} \qquad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix} \qquad \underline{v}_2 = \begin{bmatrix} c \\ 0 \\ -1 \end{bmatrix}$$

Suppose $\underline{u} = a_1 \underline{v}_1 + a_2 \underline{v}_2$

So
$$\begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ a_1 c \end{bmatrix} + \begin{bmatrix} a_2 c \\ 0 \\ -a_2 \end{bmatrix} = \begin{bmatrix} a_1 + \cancel{a_2 c} \\ 0 \\ \cancel{a_1 c} - a_2 \end{bmatrix}$$

Need $\underline{c=0}$. Take $a_1 = 1$, $a_2 = -1$.

__Exer__: Show $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a lin comb of vectors $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Suppose $\underline{u} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3$

(This is a sys of 2 eqns in 3 vars)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 + 2a_3 \\ a_2 + a_3 \end{bmatrix}$$

Write as augmented matrix:

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

RREF.

a_3 any number, $a_1 = 1 - 2a_3$
$a_2 = 1 - a_3$

<u>Def</u>. The <u>span</u> of a list of vectors $\underline{v}_1, \dots, \underline{v}_k$ is the set of all lin. combs:

~~Span~~ $\{ a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_k\underline{v}_k$ // any numbers $a_1, \dots, a_k. \}$

$\text{Span}\{\underline{v}_1, \dots, \underline{v}_k\}$

__Exer__ Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix} \right\}$

__Need__: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$ some a_1, a_2

(This is lin sys: 3 eqns, 2 vars)

$$\left[\begin{array}{cc:c} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 0 & 6 & 3 \end{array}\right]$$ ..... put in RREF

Take

__Soln__, $a_1 = 1$, $a_2 = \frac{1}{2}$.

Check!

<u>Observe</u>: vector $\underline{u}$ is lin comb of vectors $\underline{v}_1, \dots, \underline{v}_k$ with coeffs $a_1, \dots, a_k$

m-vectors

$\Updownarrow$ (if and only if)

numbers $a_1, \dots, a_k$ solve lin sys with augmented matrix

$$m\left\{\underbrace{\left[\begin{array}{cccc|c} | & | & & | & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k & \underline{u} \\ | & | & & | & | \end{array}\right]}_{k+1}\right.$$

__Notation__: We write $\mathbb{R}^n$ for the set of all n-vectors $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ with real entries.

Similarly we write $\mathbb{C}^n$ . . .

. _ _ . . . _

complex . . . . ..

Matrix notation: 3 equivalent notations

1) Lin Sys:
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ x_1 \qquad\quad - 2x_3 &= 0 \end{aligned}$$

2) Augmented matrix:
$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 7 \\ 1 & 0 & -2 & 0 \end{array}\right]$$

3) Matrix equation: $A\underline{x} = \underline{b}$

$$\underbrace{\begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 7 \\ 0 \end{bmatrix}}_{\underline{b}}$$