Lecture 18 Applications of Orthogonality

Thurs Off Hours 12-2 pm 736 Evans

Fri Quiz through 6.5

<u>Warmup!</u> Find orthog basis for soln set of $x_1 + x_2 + x_3 + x_4 = 0$

<u>Strategy</u>: 1) Find basis.

2) Orthogonalize.

1) $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

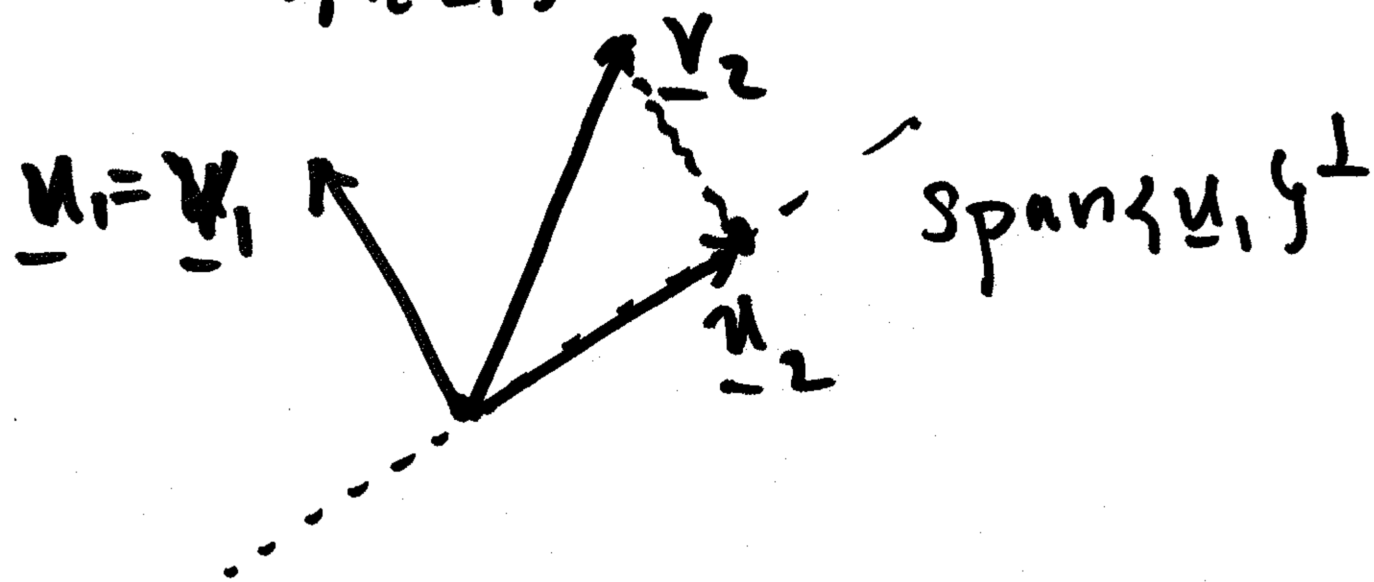
pivot free

Basis $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

2) Orthogonal basis via Gram-Schmidt

$$\underline{u}_1 = \underline{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{Span}\{\underline{u}_1\} = \mathrm{Span}\{\underline{v}_1\}$$

$$\underline{u}_2 = \mathrm{proj}_{\mathrm{Span}\{\underline{u}_1\}^{\perp}}(\underline{v}_2) = \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1$$



$$\underline{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$\mathrm{Span}\{\underline{u}_1, \underline{u}_2\} = \mathrm{Span}\{\underline{v}_1, \underline{v}_2\}$$

$$\underline{u}_3 = \mathrm{proj}_{\mathrm{Span}\{\underline{u}_1, \underline{u}_2\}^\perp}(\underline{v}_3) = \underline{v}_3 - \frac{\underline{v}_3 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\underline{u}_1 - \frac{\underline{v}_3 \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\underline{u}_2$$

$$\underline{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$= \ldots \quad \text{simplify}$$

$$\text{Span}\{\underline{u}_1, \underline{u}_2, \underline{u}_3\} = \text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$$

Warmup 2 Find orthonormal basis for image of $\underline{x} \longmapsto A\underline{x}$ where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Strategy
1) Find basis
2) Orthogonalize via G-S.
3) Normalize

1) **Basis** : colums

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

2) <u>Apply G-S</u> :

$$\underline{u}_1 = \underline{v}_1$$

$$\underline{u}_2 = \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} - \frac{(-2)}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

3) <u>Normalize</u>:

$$\hat{\underline{u}}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{\underline{u}}_2 = \frac{\underline{u}_2}{\|\underline{u}_2\|} = \frac{3}{\sqrt{6}} \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Organizing solution: $A = QR$ factorization

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} | & | \\ \hat{\underline{u}}_1 & \hat{\underline{u}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

orthonormal columns

$$R = \begin{bmatrix} \sqrt{3} & \sqrt{3}\left(-\frac{2}{3}\right) \\ 0 & \frac{\sqrt{6}}{3} \end{bmatrix} = \begin{bmatrix} \|\underline{u}_1\| & \|\underline{u}_1\| \frac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \\ 0 & \|\underline{u}_2\| \end{bmatrix}$$

upper Δ-ar
with pos. entries
on diagonal

<u>Recall</u>: $\underline{u}_2 = \underline{v}_2 - \dfrac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1$

<u>So</u>: $\underline{v}_2 = \underline{u}_2 + \dfrac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1$

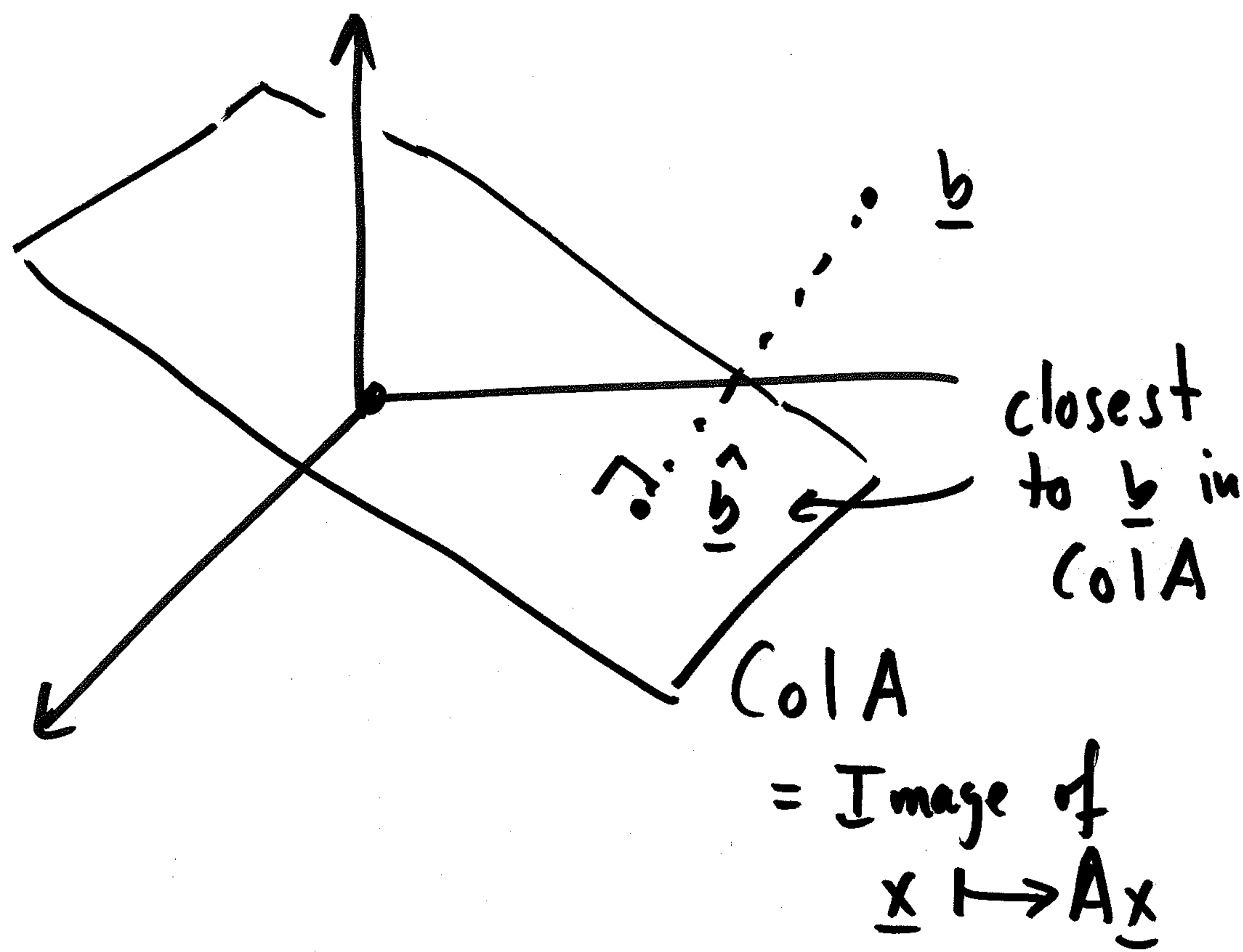
Another application of orthogonality

Least squares approximation to $A\underline{x} = \underline{b}$

If inconsistent, we can seek $\hat{\underline{x}}$ so that

$$\|\underline{b} - A\hat{\underline{x}}\| \leq \|\underline{b} - A\underline{x}\| \quad \text{for all } \underline{x}$$

Picture $\mathbb{R}^m$ = codomain



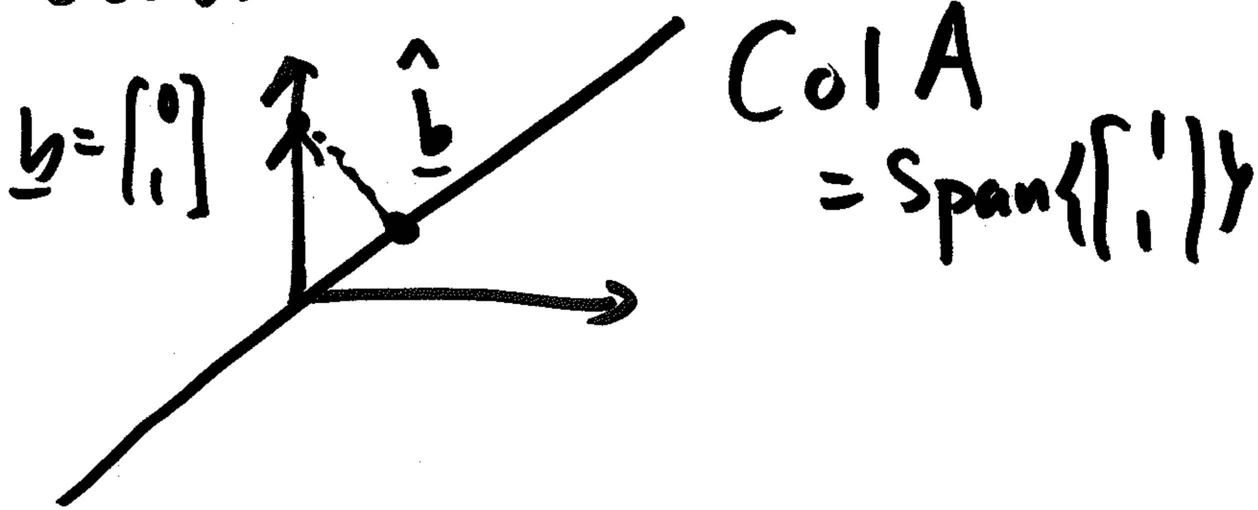
Want to solve

$$A\underline{\hat{x}} = \hat{\underline{b}} \quad \text{where} \quad \hat{\underline{b}} = \text{proj}_{\text{Col}A}(\underline{b})$$

$\underline{\text{Exer}}$ Find least square approx soln to

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\underline{\text{Picture}}$: $\mathbb{R}^2$ = codomain



$$\hat{\underline{b}} = \text{proj}_{\text{Col}A}(\underline{b}) = \frac{\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}}\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

Now solve

$$A\hat{\underline{x}} = \hat{\underline{b}} \qquad \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$\hat{x} = \begin{bmatrix}\frac{3}{2}\\1\end{bmatrix} + a\begin{bmatrix}1\\1\end{bmatrix} \qquad a \text{ any number}$$

<u>Useful general technique</u>:

Suppose $A\hat{\underline{x}} = \hat{\underline{b}}$

Observe $\underline{b} - \hat{\underline{b}} = \underline{b} - A\hat{\underline{x}}$

is $\perp$ to $\mathrm{Col}\,A$

Thus $A^T(\underline{b} - A\hat{\underline{x}}) = \underline{0}$!

since $\mathrm{Row}(A^T) = \mathrm{Col}\,A$

Arrive at formula

$$\underbrace{A^T A}_{A'} \underline{\hat{x}} = \underbrace{A^T \underline{b}}_{\underline{b}'}$$

$$A' \underline{\hat{x}} = \underline{b}'$$

<u>Exer</u> Find least sqrs approx soln to

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{\underline{b}}$$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^T \underline{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Now solve

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\underline{\hat{x}} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

Another <u>application of orthogonality</u>:

diagonalizing symmetric matrices

<u>Def</u> A $n \times n$ matrix is <u>symmetric</u>

if $A = A^T$

<u>Ex</u> $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

<u>Nonex</u> $A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

Spectral Theorem Let A be an $n \times n$ matrix.

A is symmetric if and only if

A is diagonalizable $A = PDP^{-1}$

diagonal

orthogonal matrix

<u>In other words</u>:

A admits an <u>orthormal</u> basis
of eigenvectors
if and only if
A is symmetric