Lecture 15 Diagonalization as Change of Basis

This week:

Thurs Off Hrs 12-2 pm
736 Evans

Fri Quiz through 6.1

Next week:

Tues Review 11-2
TBA

Thur Midterm 2
through 6.3

Fri No quiz

<u>Warmup!</u> Decide if

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

is 1) diagonalizable?

2) invertible?

__Recall:__ A is __diagonalizable__ if there are matrices P, D so that

invertible diagonal

$$A = PDP^{-1}$$

(Equivalently $AP = PD$)

__Step__ 1 Find e-values.

Find solns of $\chi_A(\lambda) = \det(A-\lambda)=0$.

$$A-\lambda = \begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix}$$

$$\chi_A(\lambda) = -\lambda^3 + 2\lambda^2 - \lambda$$
$$= -\lambda(\lambda^2 - 2\lambda + 1)$$
$$= -\lambda(\lambda-1)^2$$

__E-values__

$\lambda = 0, 1$

mult 1 mult 2

<u>Step</u> 2 Find e-vectors.

<u>$\lambda = 0$</u> $A - 0 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $\mathrm{Null}(A-0)$ $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\underline{\lambda = 1}$ $\quad A - 1 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for Null$(A-1)$ $\quad \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Step 3 Concatenate into single list of e-vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We found basis!

(enough e-vectors)

$n = 3$

$$D = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Remark: We saw that if $n \times n$ matrix A has n distinct e-values, then A is diagonalizable.

But it is not necessary for A to have n distinct e-values for it to be ~~invertible~~ diagonalizable!

Is A invertible? No!

<u>Reason 1</u> 0 is e-value of A.

<u>Reason 2</u> Suppose A were invertible

We've seen $A = PDP^{-1}$

So $D = P^{-1}AP$

Then D would be invertible

$D^{-1} = P^{-1}A^{-1}P$ But $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lightning$

Warmup 2 Show $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

1) not diagonalizable.
2) invertible. $A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

Step 1 Find e-values

$\lambda = 2$.

mult $= 2$

— A is Δ-ar so diag entries are e-values

— $\chi_A(\lambda) = (2-\lambda)^2$

__Step__ 2 Find e-vectors

$$A - 2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \dim \operatorname{Nul}(A-2) = 1$$

$$\text{Basis } \underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Not __diagonalizable__ : not enough e-vectors.

Let's see what it would mean if

$$A = PDP^{-1}, \quad D \text{ diagonal}$$

<u>Exer</u> Show cols of P are e-vectors for A, using eqn $A = PDP^{-1}$ and that D is diagonal

So cols of P would form a basis of e-vectors.

__Challenge__ What's the closest we can get to diagonalizing $A = \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}$?

$$\chi_A(\lambda) = \det(A-\lambda)$$
$$= (1-\lambda)(3-\lambda)+2$$
$$= \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$\lambda_+ = 2+i$, $\lambda_- = 2-i$

Re $= 2$ Im $= \pm 1$

If we insist on only real matrices, A is <u>not</u> diagonalizable.

But if we allow complex numbers we should take $D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$

(Always find $\lambda_- = \overline{\lambda_+}$)

Find complex e-vectors $\underline{v}_+, \underline{v}_-$

$$A - \lambda_+ = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}$$

We know there is nontriv. null space

$$\underline{v}_+ = \begin{bmatrix} -2 \\ 1+i \end{bmatrix}$$

<u>Check:</u>

$(A - \lambda_+)\underline{v}_+ = \underline{0}$.

Similarly ... find $\underline{v}_-$

$$A - \lambda_- = \begin{pmatrix} -1+i & -2 \\ 1 & 1+i \end{pmatrix}$$

$$\underline{v}_- = \begin{bmatrix} -2 \\ 1-i \end{bmatrix}$$

<u>Check:</u>

$$(A - \lambda_-)\underline{v}_- = \underline{0}$$

(Always take $\underline{v}_- = \overline{\underline{v}_+}$)

What if we're Neanderthals and don't accept complex numbers?

<u>Best alternative:</u> Find P, C

invertible

rotation + scaling

so that

$$A = PCP^{-1}$$

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$r = \sqrt{a^2+b^2}$
$a = r\cos\theta$
$b = r\sin\theta$

Back to challenge problem:

$$\lambda_+ = \underset{\substack{\uparrow \\ a}}{2} + i\cdot\underset{\substack{\uparrow \\ -b}}{1} \qquad \underline{v}_+ = \begin{bmatrix} -2 \\ 1+i \end{bmatrix}$$

Take

$$C = \begin{bmatrix} 2 & +1 \\ -1 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} | & | \\ \mathrm{Re}(\underline{v}_+) & \mathrm{Im}(\underline{v}_+) \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$$

<u>Note</u>

$$C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\text{scale}} \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & +\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}_{\text{rotation}}$$

Conceptual meaning of diagonalization

What does $A = PDP^{-1}$ mean?

<u>Def</u>. $n \times n$ matrices A, A' are <u>similar</u> if there is an invertible $n \times n$ matrix P so that

$$A = PA'P^{-1}$$

You should think of similar matrices as "the same transf" from different perspectives.

Let's think of the cols of P as a basis $\mathcal{B} = \{\underline{v}_1, \ldots, \underline{v}_n\}$

Then $P = P_{\mathcal{B}} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$

$$\underline{x} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}}$$

And $P^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$

$$[\underline{x}]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \underline{x}$$

So $A = PA'P^{-1} = \underset{\mathcal{E} \leftarrow B}{P} A' \underset{B \leftarrow \mathcal{E}}{P}$

<u>Conclusion</u>: A' is matrix of A with respect B coords

$$[A\underline{x}]_B = A'[\underline{x}]_B$$