# Lecture 13 The Fun Begins!

Eigenvectors and Eigenvalues

Thurs Off. Hours 12-2pm 736 Evans

Fri Quiz thru 5.2.

__Warmup__ Suppose coords of vector $\underline{x}$ in $\mathbb{R}^3$ are $\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ w.r.t. basis $B = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \right\}$

Find basis $C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \underline{c}_2, \underline{c}_3 \right\}$ s.t.

coords of $\underline{x}$ are $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ w.r.t. $C$.

__Soln__: $\underline{x} = 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} + (-3) \cdot \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

$$\underline{x} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1 \cdot \underline{c}_2 + 3\underline{c}_3$$

$$\begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = \underline{c}_2 + 3\underline{c}_3$$

Lin syst in 6 vars
3 eqns

$$\begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\underline{c}_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \underline{c}_3 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\left.\begin{array}{l}\text{y's are pivot vars}\\ \text{z's are free vars}\end{array}\right\}\quad \underline{\text{But}}\text{ need } \underline{c}_1, \underline{c}_2, \underline{c}_3 \text{ to be basis!}$$

$$\underline{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \text{Choose } \underline{c}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Then } \underline{c}_2 = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$$

$$\text{Basis } C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Now onto Eigen<u>vectors</u> & Eigen<u>values</u>

We are world's experts on lin systs

$$A \underline{x} = \underline{b}$$

$m \times n$ matrix

$n$ vector of vars

$m$ vector of values.

# Second Great Eqn of Lin Alg

$$A \underline{x} = \lambda \underline{x}$$

n x n matrix (square!)

number

n vector of vars.

$\triangle$ 2 $\underline{x} = \underline{0}$ always is a soln for any $\lambda$

So we will <u>never</u> consider $\underline{x} = \underline{0}$.

<u>Note</u> 1) If $\underline{x}$ solves $A\underline{x} = \lambda \underline{x}$

and $A\underline{x} = \mu \underline{x}$

then $\lambda \underline{x} = A\underline{x} = \mu \underline{x}$

Since $\underline{x} \neq \underline{0}$, we have $\lambda = \mu$.

So an eigenvector $\underline{x}$ has unique eigenvalue $\lambda$.

2) If $\lambda$ solves $A\underline{x} = \lambda \underline{x}$

there can be many such $\underline{x}$.

<u>Example</u>: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Any vector $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \underline{0}$ is an e-vector for $\lambda = 2$.

Suppose some $\lambda$ (possibly $= 0$) solve
some $\underline{x}$ ( $\underline{\underline{\text{not}}} = \underline{0}$ )

$$A \underline{x} = \lambda \underline{x}$$

eigenvalue of A

eigenvector with eigenvalue $\lambda$

__Observe__ If $\lambda = 0$, back to solving

$$A\underline{x} = 0 \cdot \underline{x} = \underline{0}$$

$$\left\{ \begin{array}{c} \text{Eigenvectors} \\ \text{for } \lambda = 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Null space} \\ \text{of } A \end{array} \right\}$$

(together with $\underline{x} = \underline{0}$)

In general

$$A\underline{x} = \lambda \underline{x} \iff (A - \lambda I_n)\underline{x} = \underline{0}$$

$$\left\{ \begin{array}{c} \text{Eigenvectors} \\ \text{for } \lambda \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Null space} \\ \text{of } A - \lambda I_n \end{array} \right\}$$

(together with $\underline{x} = \underline{0}$)

**Terminology:** Eigenspace for $\lambda$ is Null space of $A - \lambda I_n$

**Example**

1) $A = \begin{bmatrix} 7 & & & & 0 \\ & -1 & & & \\ & & 2 & & \\ & & & 3 & \\ 0 & & & & -1 & \\ & & & & & 0 \end{bmatrix}$ diagonal matrix

List all e-values and e-vectors.

E-values of diag matrix are diag entries: $7, -1, 2, 3, 0$

for example

$$(A - 2I_6) = \begin{bmatrix} 5 & & & & & \\ & -3 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & -3 & \\ & & & & & -2 \end{bmatrix}$$

$$\text{Null space} = \text{Span}\left\{ \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

This is eigenspace for $\lambda = 2$.

<u>Note</u>: eigenspace is 1 dim

for example

$$(A - (-1)I_6) = \begin{bmatrix} 8 & & & & & \\ & 0 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix}$$

Null space $= \text{span}\{\underline{e}_2, \underline{e}_5\}$

This is eigenspace for

$\lambda = -1$

<u>Note</u>: eigenspace is 2 dim.

but

$$(A - 4I_6) = \begin{pmatrix} 3 & & & & & \\ & -5 & & & & \\ & & -2 & & & \\ & & & -1 & & \\ & & & & -5 & \\ & & & & & -4 \end{pmatrix}$$

Here null space $= \{ \underline{0} \}$

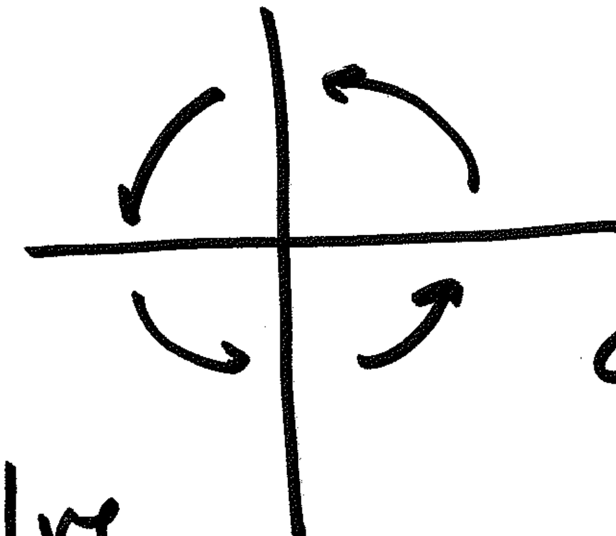
So 4 is <u>not</u> an eigenvalue.

## $2 \times 2$ examples

1) $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$    e-values $0, -3$

e-vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



rotates $\frac{\pi}{2}$ rads c.cl.wise

Impossible to solve

$A\underline{x} = \lambda \underline{x}$ for any $\lambda$

if $\underline{x} \neq \underline{0}$

Algebraic reason: Suppose $\lambda$ is a possible e-value

Want to find null space of

$$(A - \lambda I_2) = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda I_2) = \lambda^2 + 1 \neq 0$$

never $\}$ for a real number

So $A - \lambda I_2$ always invt-ble!

3) $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

<u>e-values</u>: 3

<u>e-space for $\lambda = 3$:</u>

$\text{span}\left\{ \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Find null space of $(A - \lambda I_2) = \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I_2) = (3-\lambda)^2$

only $= 0$ when $\lambda = 3$.

So $A - \lambda I_2$ has nontriv null space only if $\lambda = 3$.

When $\lambda = 3$

$$A - \lambda I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{Null space} = \operatorname{span}\left\{ \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

4) Find e-values and e-spaces

of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Consider $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}$

When does this have nontriv null space?

Not invt-ble, or equivalently

$\det(A - \lambda I_2) = 0$.

$$\begin{aligned} \det(A - \lambda I_2) &= (1-\lambda)(2-\lambda) - 30 \\ &= 2 - 3\lambda + \lambda^2 - 30 \\ &= \lambda^2 - 3\lambda - 28 \\ &= (\lambda - 7)(\lambda + 4) \end{aligned}$$

E-values: $\lambda = 7, -4$

Find e-spaces!