Welcome to the Age of Abstraction!

(Also known as Lecture 10)

<u>Midterm 1 Review</u>: during off. hours

Thurs. 12 - 2   736 Evans

(Note unusual time!)

<u>Also</u>: see webpage for GSI reviews

<u>Fri</u> Quiz through 4.4

<u>Warmup</u>: $V = \mathbb{P}_2$, $W = \mathbb{R}^2$

Define $T: V \longrightarrow W$

$$T(p) = \begin{bmatrix} p(2) \\ \frac{dp}{dx}(-1) \end{bmatrix}$$

1) Show T is lin. transf.

<u>Exer</u>. Check 2 axioms!

2) Find eqns for Nul T.

$T(p) = \underline{0}$?

Need $p(2) = 0$ and $\frac{dp}{dx}(-1) = 0$

If $p(x) = a_0 + a_1 x + a_2 x^2$

then need : $a_0 + 2a_1 + 4a_2 = 0$

$a_1 + 2a_2 \cdot (-1) = 0$

<u>Eqns</u> : $a_0 + 2a_1 + 4a_2 = 0$

$a_1 - 2a_2 = 0$

3) Find spanning list for Range T.

$$\left.\begin{aligned} T(1) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{v}_1 \\ T(x) &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \underline{v}_2 \end{aligned}\right\} \text{ already span all of } \mathbb{R}^2$$

So they span Range T.

Today: continue with abstract versions of old ideas.

Last time – $\mathbb{R}^n \rightsquigarrow V$ vect. sp.

– $T:\mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow T: V \to W$

given by $m \times n$ matrix A — lin. transf.

– subsp. H in $\mathbb{R}^n \rightsquigarrow$ subsp H in V

(Nul A, Col A, ...) — (Nul T, Range T, ...)

__Def__ Let V be a vect. sp.
Let $\underline{v}_1, \dots, \underline{v}_k$ be a list of vects. in V
We say $\underline{v}_1, \dots, \underline{v}_k$ is __lin indep.__ if

$$a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k = 0 \quad \text{implies}$$

$$a_1 = \cdots = a_k = 0$$

__Def__ Let V be a vect. sp., H in V a subsp.
List $\underline{v}_1, \dots, \underline{v}_k$ in H is a __basis__
if
1) $\mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_k\} = H$
2) $\underline{v}_1, \dots, \underline{v}_k$ lin. indep.

__Ex__ 1) $V = \mathbb{R}^n$, standard basis

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

2) $V = \mathbb{P}_n$, stand. basis

$$1, \ x, \ x^2, \ \dots, \ x^n$$

Exer $T : \mathbb{P}_3 \longrightarrow \mathbb{R}^1$

$$T(p) = p(1)$$

Find basis for Nul T.

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_0 + a_1 + a_2 + a_3$$

Strategy: 1) Find enough vectors to span Nul T

2) Throw out unnec. ones from list

lin comb of others

1) $\underline{v}_1 = 1 - x$, $\underline{v}_2 = 1 - x^2$, $\underline{v}_3 = 1 - x^3$,

$\underline{v}_4 = x - x^2$, $\underline{v}_5 = x^2 - x^3$

<u>Check</u>: $\operatorname{Span}\{\underline{v}_1, \ldots, \underline{v}_5\} = NUIT$

2) $\underline{v}_1 = 1 - x$, $\underline{v}_2 = 1 - x^2$, $\underline{v}_3 = 1 - x^3$

But $\underline{v}_4 = (-1)\underline{v}_1 + \underline{v}_2$

<u>Check</u>: $\underline{v}_5$ lin comb of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

So basis $\underline{v}_1 = 1 - x$, $\underline{v}_2 = 1 - x^2$, $\underline{v}_3 = 1 - x^3$

<u>Exer</u> Find basis for Range T where

$$T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2 \qquad T(p) = 2\frac{dp}{dx} - p(0)$$

$$T(a_0 + a_1 x + a_2 x^2) = 2(a_1 + 2a_2 x) - a_0$$

$$= (2a_1 - a_0) + 4a_2 \cdot x$$

<u>Use same strategy</u>:

1) $\left.\begin{aligned} T(1) &= -1 = \underline{v}_1 \\ T(x) &= 2 = \underline{v}_2 \\ T(x^2) &= 4x = \underline{v}_3 \end{aligned}\right\}$ Span Range T b/c $1, x, x^2$ span domain

2) $\underline{v}_1 = -1$, $\underline{v}_3 = 4x$

Basis is $\underline{v}_1, \underline{v}_3$.

Why do we like bases?

Thm V vect sp, $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ basis of V

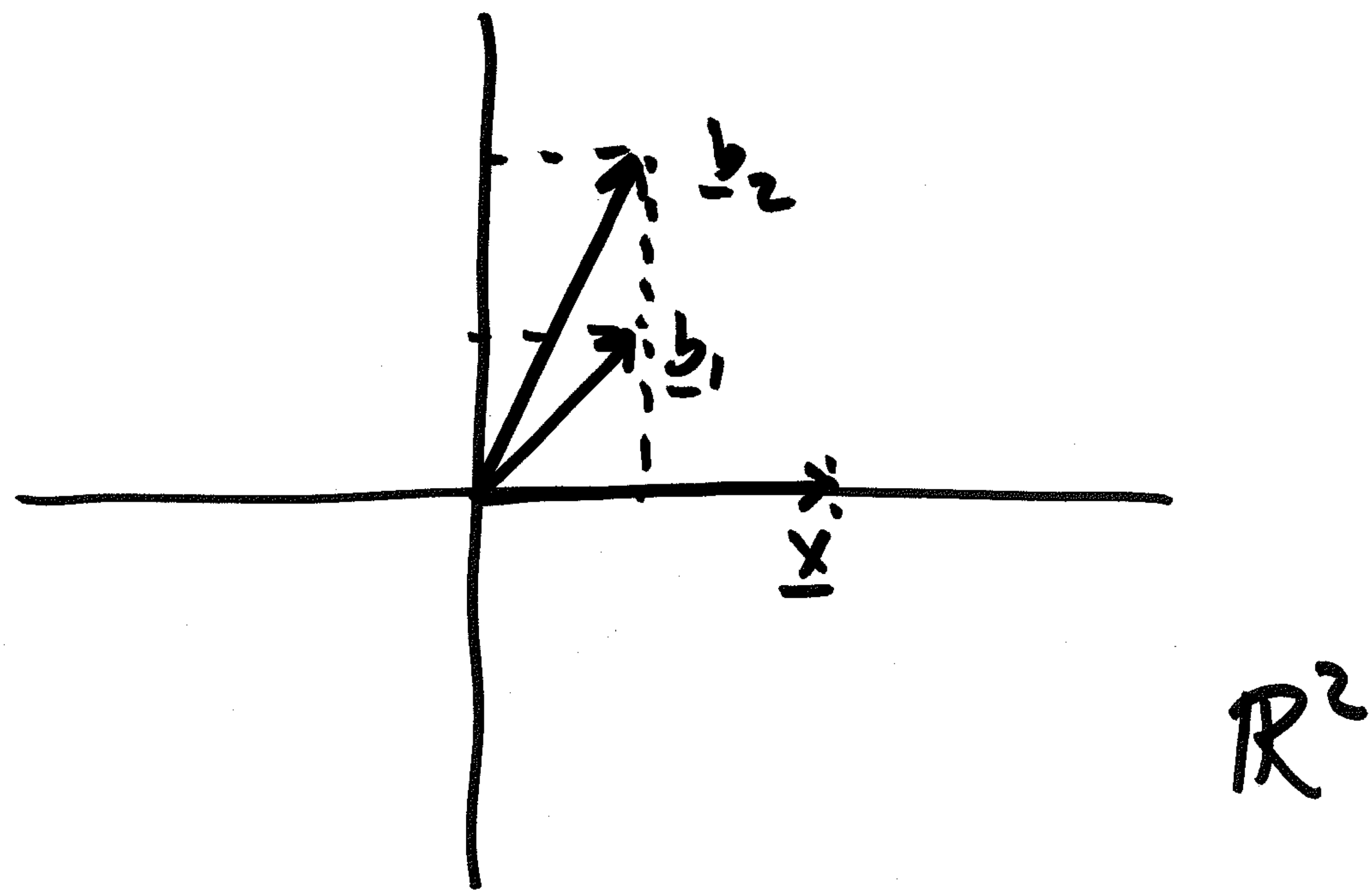
Then each $\underline{x}$ in V can be uniquely written as lin comb

$$\underline{x} = c_1 \underline{b}_1 + \cdots + c_n \underline{b}_n$$

Organize into a coord vect. $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

<u>Exer</u> $V = \mathbb{R}^2$, $B = \left\{ \underline{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \underline{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Find coord vect $[\underline{x}]_B$ where $\underline{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$



Find $\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{= [\underline{x}]_B}$ so that $\underline{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solve $\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\underline{x}} = \overbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}^{P_B = } \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{[\underline{x}]_B}$

$\underline{b}_1 \quad \underline{b}_2$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{= P_B^{-1}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = [\underline{x}]_B$$

General story If $B = \{\underline{b}_1, \ldots, \underline{b}_n\}$ is basis for $\mathbb{R}^n$ then coord vect. of $\underline{x}$ in $\mathbb{R}^n$ satisfies

$$\underline{x} = \underbrace{\begin{bmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_n \\ | & & | \end{bmatrix}}_{= P_B} [\underline{x}]_B$$

So to find coords, we use

$$[\underline{x}]_B = P_B^{-1}\,\underline{x}$$

<u>Exer</u> Find coords of $\underline{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ w.r.t. basis $B = \left\{ \underline{b}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \underline{b}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \underline{b}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

$$\underline{x} = \underbrace{\begin{bmatrix} | & | & | \\ \underline{b}_1 & \underline{b}_2 & \underline{b}_3 \\ | & | & | \end{bmatrix}}_{=P_B} [\underline{x}]_B \quad \text{so} \quad [\underline{x}]_B = P_B^{-1}\,\underline{x}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Now solve system!

Choices: 1) Find P_B^{-1}

2) Row reduction

<u>Soln:</u> $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = [\underline{x}]_B$