Chapter 6

1. *Proof.* In this problem we use $\langle \cdot, \cdot \rangle$ to denote the standard Euclidean inner product.

As the hint in the book suggests, we consider the triangle with sides x, y, and $x - y$. Let θ denote the angle from vector x to vector y, again in the standard Euclidean sense. Since cos is an even function, it doesn't matter whether the angle θ going from x to y is inside the triangle or the angle $-\theta$ from y to x is inside; they have the same cosine. The law of cosines states that

$$\|x\|^2 + \|y\|^2 = \|x - y\|^2 - 2\|x\|\|y\| \cos \theta.$$

We can expand out

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle.$$

Substituting back in,

$$\|x\|^2 + \|y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle - 2\|x\|\|y\| \cos \theta.$$

Thus, $\langle x, y \rangle = \|x\|\|y\| \cos \theta$ as desired. □

2. *Proof.* $\Rightarrow$: If $\langle u, v \rangle = 0$ then for any $a \in \mathbf{F}$,

$$\|u + av\|^2 = \|u\|^2 + |a|^2\|v\|^2 \geq \|u\|^2,$$

as desired.

$\Leftarrow$: We proceed by contraposition. Assume that $\langle u, v \rangle \neq 0$ and we will show that for some $a \in \mathbf{F}$,

$$\|u\| > \|u + av\|.$$

In particular, we will show that

$$\|u + av\|^2 - \|u\|^2 < 0.$$

We begin by expanding.

$$\begin{aligned} \|u + av\|^2 - \|u\|^2 &= |a|^2\|v\|^2 + \langle u, av \rangle + \langle av, u \rangle \\ &= |a|^2\|v\|^2 + 2\text{Re}(a\langle v, u \rangle) \end{aligned}$$

By our premise, $\langle v, u \rangle = re^{i\theta}$ for some unique $r > 0$ and $\theta \in [0, 2\pi)$. We will take a to be of the form $a = -r'e^{-i\theta}$ for some $r' > 0$. So

$$\begin{aligned} \|u + av\|^2 - \|u\|^2 &= (r')^2\|v\|^2 + 2\text{Re}(-r'e^{-i\theta}re^{i\theta}) \\ &= (r')^2\|v\|^2 - 2r'r. \end{aligned}$$

By our premise, $\|v\| > 0$. We want

$$\begin{aligned} (r')^2\|v\|^2 - 2r'r &< 0 \\ r'(r'\|v\|^2 - 2r) &< 0 \\ r'\|v\|^2 - 2r &< 0 \\ r' &< \frac{2r}{\|v\|^2}. \end{aligned}$$

So $r' = \frac{r}{\|v\|^2}$ gives the desired result. That is to say, we take

$$a = -\frac{re^{-i\theta}}{\|v\|^2} = -\frac{\overline{\langle u, v\rangle}}{\|v\|^2}.$$

□

3. *Proof.* Take

$$u = (a_1, a_2\sqrt{2}, a_3\sqrt{3}, \cdots, a_n\sqrt{n}), \text{ and}$$
$$v = \left(b_1, \frac{b_2}{\sqrt{2}}, \frac{b_3}{\sqrt{3}}, \cdots, \frac{a_n}{\sqrt{n}}\right)$$

By Cauchy-Schwartz, applied to the dot product on $\mathbf{R}^n$,

$$\sum_{j=1}^{n} a_j b_j = (u \cdot v)^2 \leq \|u\|^2\|v\|^2 = \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right).$$

□

4. We begin by squaring

$$\begin{aligned} 36 + 16 = \|u+v\|^2 + \|u-v\|^2 &= 2\|u\|^2 + 2\|v\|^2 + 2\mathrm{Re}(\langle u, v\rangle) - 2\mathrm{Re}(\langle u, v\rangle) \\ &= 18 + 2\|v\|^2. \end{aligned}$$

Thus $2\|v\|^2 = 34$, so $\|v\| = \sqrt{17}$.

5. There is no such inner product.

Proof. Suppose for a contradiction that some such inner product $\langle \cdot, \cdot \rangle$ exists. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle.$$

Here we are appealing to the symmetry of inner products on real vector spaces. For $u = (u_1, u_2)$ and $v = (v_1, v_2)$ elements of $\mathbf{R}^2$, this gives the formula

$$\langle u, v\rangle = \frac{1}{2}((|u_1 + v_1| + |u_2 + v_2|)^2 - (|u_1| + |u_2|)^2 - (|v_1| + |v_2|)^2).$$

Plugging in, $\langle (0,1), (1,0)\rangle = \langle (0,-1), (0,1)\rangle = 1$. This violates homogeneity. □

6. *Proof.* We verify this by computation.

$$\begin{aligned} \|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\ &= (\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle) \\ &\quad - (\langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle) \\ &= 2\langle u, v\rangle + 2\langle v, u\rangle \\ &= 4\langle u, v\rangle. \end{aligned}$$

On the last line we are appealing to the symmetry of the inner product on a real vector space. □

7. *Proof.* As in the previous problem, we verify this by computation. As before,

$$\|u+v\|^2 - \|u-v\|^2 = 2\langle u, v\rangle + 2\langle v, u\rangle.$$

But now, appealing to skew-symmetry,

$$\|u+v\|^2 - \|u-v\|^2 = 2\langle u, v\rangle + 2\overline{\langle u, v\rangle} = 4\text{Re}(\langle u, v\rangle).$$

Likewise,

$$\begin{aligned} i\|u+iv\|^2 - i\|u-iv\|^2 &= 2i\langle u, iv\rangle + 2i\langle iv, u\rangle \\ &= 2i(-i)\langle u, v\rangle + 2i^2\langle v, u\rangle \\ &= 2\langle u, v\rangle - 2\overline{\langle u, v\rangle} \\ &= 4i\text{Im}(\langle u, v\rangle). \end{aligned}$$

Thus,

$$\begin{aligned} \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2 &= 4\text{Re}(\langle u, v\rangle) + 4i\text{Im}(\langle u, v\rangle) \\ &= 4\langle u, v\rangle. \end{aligned}$$

□