Solutions to Homework #7.

Chapter 5.

13. To make things concrete, let's choose a basis $v_1, \dots, v_n$ of V so that T is represented by a matrix $[T]$ with entries denoted T_{ij}.

Consider the subspace $W = span(v_2, \dots, v_n) \subset V$ and note $\dim W = n-1$. Thus W is T-invariant by assumption. Thus for any $i = 2, \dots, n$, we have $Tv_i \in W$ and so we have $T_{i1} = 0$. In other words the first row of $[T]$ is all zeros except for possibly T_{11}.

Next take $W = span(v_1, v_3, \dots, v_n) \subset V$ and note $\dim W = n-1$. Thus W is T-invariant by assumption. Thus for any $i = 1, 3, \dots, n$, we have $Tv_i \in W$ and so we have $T_{i2} = 0$. In other words the second row of $[T]$ is all zeros except for possibly T_{22}.

Continuing in this way we see that $[T]$ must be diagonal.

Now choose $W = span(v_1 + v_2, v_3, \dots, v_n) \subset V$ and note $\dim W = n-1$ since the $n-1$ vectors $v_1 + v_2, v_3, \dots, v_n$ are linearly independent. Thus W is T-invariant by assumption. Thus $T(v_1 + v_2) = T_{11}v_1 + T_{22}v_2 \in W$ which implies $T_{11}v_1 + T_{22}v_2 = a(v_1 + v_2)$ for some $a \in F$. Hence $(T_{11} - a)v_1 + (T_{22} - a)v_2 = 0$ and by linear independence we see $a = T_{11} = T_{22}$.

Next choose $W = span(v_1, v_2 + v_3, v_4, \dots, v_n) \subset V$ and note $\dim W = n-1$ since the $n-1$ vectors $v_1, v_2 + v_3, v_4, \dots, v_n$ are linearly independent. Thus W is T-invariant by assumption. Thus $T(v_2 + v_3) = T_{22}v_2 + T_{33}v_2 \in W$ which implies $T_{22}v_1 + T_{33}v_2 = a(v_1 + v_2)$ for some $a \in F$. Hence $(T_{22} - a)v_1 + (T_{33} - a)v_2 = 0$ and by linear independence we see $a = T_{22} = T_{33}$.

Continuing in this way we see that $[T]$ must be of the form $a \, \mathrm{Id}_V$.

14. If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then $p(STS^{-1}) = a_0 + a_1(STS^{-1}) + \cdots + a_n(STS^{-1})^n = a_0 + a_1(STS^{-1}) + \cdots + a_n(ST^nS^{-1}) = S(a_0 + a_1 T + \cdots + a_n T^n)S^{-1} = Sp(T)S^{-1}$.

15. Choose a basis of V so that $[T]$ is upper triangular. We know that its eigenvalues are its diagonal entries λ_i. Then with respect to the same basis, we see that $[p(T)]$ will also be upper triangular with diagonal entries $p(\lambda_i)$. Thus a is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T.

16. Counterexample: take $V = \mathbb{R}^2$,

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $p(z) = z^2$. Then T has no eigenvalues but $p(T) = T^2 = -\mathrm{Id}_V$ has eigenvalue -1.

17. Choose basis $v_1, \dots, v_n$ of V such that $[T]$ is upper triangular. Then we have proved (see Proposition 5.12) that $V_j = span(v_1, \dots, v_j)$ is T-invariant and clearly $\dim V_j = j$.

18. Take $V = \mathbb{R}^2$ and

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then T is invertible with $T^{-1} = -T$.

19. Take $V = \mathbb{R}^2$ and

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Then T is not invertible since $null\, T$ contains $(1, -1)$.

20. Let $n = \dim V$. Since T has n distinct eigenvalues, there is a linearly independent set $v_1, \ldots, v_n$ of eigenvectors and hence a basis. Thus $[T]$ becomes diagonal but so does $[S]$. Thus $ST = TS$.

21. Take $v \in V$. Consider $w = v - Pv$. Then $Pw = Pv = P^2v = 0$ so $w \in null\, P$. Writing $v = w + Pv$ we see $V = null\, P + range\, P$. Thus it suffices to see $null\, P \cap range\, P = \{0\}$. Take $u \in null\, P$ and $Pv \in range\, P$ and suppose $w = Pv$. Then $0 = Pw = P^2v = Pv = w$. So we see w and Pv are 0 as desired.

22. Any $u \in U$ satisfies $P_{U,W}u = u$ and hence if $u \neq 0$, it is an eigenvector of eigenvalue 1.

Any $w \in W$ satisfies $P_{U,W}w = 0$ and hence if $w \neq 0$, it is an eigenvector of eigenvalue 0.

If any other vector $v = u + w \neq 0$ satisfies $P(u + w) = a(u + w)$ then we have $u = P(u + w) = a(u + w)$ so either $w = 0$ and $a = 1$, or $a = 0$ and $u = 0$.

23. Take

$$T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then $T^2 = -\operatorname{Id}$. Thus if $Tv = av$, with $v \neq 0$, $a \in \mathbb{R}$, then $-v = -\operatorname{Id} v = T^2v = aTv = a^2v$, and so $a^2 = -1$, a contradiction.

24. This follows from Theorem 5.26. Namely, if there is a T-invariant subspace $W \subset V$ of odd dimension then W contains an eigenvector of T. And we can also regard this eigenvector as an eigenvector in V, a contradiction.