MATH 110 HOMEWORK 3 SOLUTIONS

1. Let $\{v\}$ be a basis for our given 1-dimensional vector space V. Since V is one-dimensional, we can write $T(v) = av$ for some scalar $a \in F$. Then for any other vector $u \in V$ we can write $u = bv$ for some $b \in F$. Then we have

$$\begin{aligned} T(u) &= T(bv) \\ &= bT(v) \\ &= bav \\ &= abv = au. \end{aligned}$$

Therefore T is multiplication by the scalar a.

2. Intuitively, the problem is asking for a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is linear when restricted to every line through the origin, but the linear functions for various lines through the origin don't fit together to give a linear function $\mathbb{R}^2 \to \mathbb{R}$. An example is given by the function

$$f(x,y) = \begin{cases} x & \text{if } \ y = 0 \\ 0 & \text{if } \ y \neq 0. \end{cases}$$

Then if $y = 0$, $f(ax, ay) = ax = af(x,y)$ and if $y \neq 0$, then $f(ax, ay) = 0 = af(x,y)$. So f is homogeneous. But f is not linear since (for example) $f(1,1) = 0 \neq 1 = f(1,0) + f(0,1)$.

3. Let U be the given subspace of V. Choose a basis $\{u_1, ..., u_n\}$ of U, and extend it to a basis $\{u_1, ..., u_n, v_1, ..., v_k\}$ of V. Given a linear transformation $S : U \to W$, define $T : V \to W$ by first defining the values of T on the basis vectors as follows:

$$\begin{aligned} T(u_i) &= S(u_i), \quad 1 \leq i \leq n, \\ T(v_i) &= 0, \quad 1 \leq i \leq k. \end{aligned}$$

As explained on p. 40 in your text, there exists a unique linear transformation $T : V \to W$ extending this definition on the basis vectors $\{u_1, ..., u_n, v_1, .., v_k\}$. Then for any vector $u \in U$, we can write $u = \sum_i a_i u_i$, and

$$T(u) = T(\sum_i a_i u_i) = \sum_i a_i T(u_i) = \sum_i a_i S(u_i) = S(u).$$

Therefore T agrees with S on U as was to be shown.

4. Let U be the vector space $\{au : a \in F\}$. We want to show that $V = \text{null}\, T \oplus U$. First we show that $\text{null}\, T \cap U = \{0\}$. Suppose $v \in \text{null}\, T \cap U$. Therefore $v = au$ for $a \in F$, and $0 = T(v) = T(au) = aT(u)$. Since $T(u) \neq 0$, this implies $a = 0$ and therefore $v = 0$.

Next we show that $V = \text{null}\, T + U$. This means that for any $v \in V$ we want to write $v = n + au$ for $n \in \text{null}\, T$ and $a \in F$. Applying T to both sides of this equation, we see that we want $T(v) = aT(u)$. Since $T(u) \neq 0$ is a non-zero scalar, we can divide to obtain $a = T(v)/T(u)$. We then set $n = v - au$, where $a = T(v)/T(u)$. We check that

$$T(n) = T(v) - \frac{T(v)}{T(u)} T(u) = 0,$$

and therefore $n \in$ null T as is required. It is clear that $v = n + au$ for this definition of n and a, showing that $V =$ null $T + U$ and completing the proof.

5. Suppose we have a linear relation $a_1T(v_1) + ... + a_nT(v_n) = 0$. By linearity, we rewrite this as $T(a_1v_1 + ... + a_nv_n) = 0$. Since T is injective, we conclude that $a_1v_1 + ... + a_nv_n = 0$, and since $\{v_1, ..., v_n\}$ is linearly independent we conclude that $a_1 = ... = a_n = 0$. Therefore $\{T(v_1), ..., T(v_n)\}$ is linearly independent.

7. Let w be any vector in W. Since $T : V \to W$ is surjective, we can write $w = T(v)$ for some $v \in V$. Since $\{v_1, ..., v_n\}$ spans V, we can write $v = a_1v_1 + ... + a_nv_n$ for some scalars $a_1, .., a_n \in F$. Therefore

$$w = T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n),$$

showing that w is in the span of $T(v_1), ..., T(v_n)$. Therefore $\{T(v_1), ..., T(v_n)\}$ spans W.

8. Let $T : V \to W$ be the given linear transformation. Let $\{n_1, ..., n_k\}$ be a basis for nulll T, and extend this to a basis $\{n_1, ..., n_k, v_1, ..., v_m\}$ of V. Let $U = \text{Span}(v_1, ..., v_m)$. We claim that this U satisfies the requirements of the proposition.

First we show that $U \cap \text{Null } T = 0$. Suppose $u \in U \cap \text{null } T$. Then

$$u = a_1n_1 + ... + a_kn_k = b_1v_1 + ... + b_mv_m$$

for some scalars $a_1, ..., a_k, b_1, ..., b_m$ (since $n_1, ..., n_k$ span nulll T and $v_1, ..., v_m$ span U). But since $\{n_1, ..., n_k, v_1, ..., v_m\}$ is a basis for V, u must have a unique expression as a linear combination of $\{n_1, ..., n_k, v_1, ..., v_m\}$. The only way for the two expressions above to be the same linear combination of $\{n_1, ..., n_k, v_1, ..., v_m\}$ is for all the coefficients to be 0, which implies that $u = 0$.

Now we must show that range $T = \{Tu : u \in U\}$. It is clear that $\{Tu : u \in U\} \subseteq$ range T, so we must show that range $T \subseteq \{Tu : u \in U\}$. Any vector in range T can be written as $T(v)$, for $v \in V$. By the definition of U we can write $v = n + u$, for $n \in$ Null T and $u \in U$. Namely, write $v = a_1n_1 + ... + a_kn_k + b_1v_1 + ... + b_mv_m$ and let $n = a_1n_1 + ... + a_kn_k$ and $u = b_1v_1 + ... + b_mv_m$. Then applying T, we get that

$$T(v) = T(n + u) = T(n) + T(u) = T(u),$$

showing that range $T \subset \{Tu : u \in U\}$ as required.

9. Notice that the dimension of Null T is 2. Namely, we have a basis for Null T consisting of the vectors $(5, 1, 0, 0)$ and $(0, 0, 7, 1)$. We obtained these vectors by plugging the values 1,0 and 0,1 respectively for the free parameters x_2 and x_4. By Theorem 3.4, dim range $T = 4 - 2 = 2$, so T must be surjective by Proposition 2.17.

10. From the given description of the null space we see that the null space has dimension 2, since there are two free parameters x_2 and x_5. But we must have

$$\text{dim V} = \text{dim Null } T + \text{dim Range } T.$$

The terms on the right hand side are at most 2, while the left hand side is 5. This is impossible.

11. We cannot directly apply Theorem 3.4 since that theorem assumes V is finite dimensional. Instead, we proceed as follows. Let $T : V \to W$ be a linear transformation with finite-dimensional null space and range. Let $\{n_1, ..., n_k\}$ be a basis for Null T and $\{w_1, ..., w_m\}$ a basis for Range T. For each w_i, let v_i be a vector in V such that $T(v_i) = w_i$. We claim that $\{n_1, ..., n_k, v_1, .., v_m\}$ spans

V, which implies that V is finite dimensional. Given any $v \in V$, we can write

$$T(v) = a_1 w_1 + ... + a_m w_m = a_1 T(v_1) + ... + a_m T(v_m).$$

Now notice that

$$T(v - a_1 v_1 - ... - a_m v_m) = T(v) - a_1 T(v_1) - ... - a_m T(v_m) = 0$$

by the above. Therefore $v - a_1 v_1 - ... - a_m v_m \in \text{Null } T$, so we can write

$$v - a_1 v_1 - ... - a_m v_n = b_1 n_1 + ... + b_k n_k$$

for some scalars $b_1, ..., b_k$. Rearranging, we have that

$$v = b_1 n_1 + ... + b_k n_k + a_1 v_1 + ... + a_m v_m$$

showing that $\{n_1, ..., n_k, v_1, ..., v_m\}$ spans V and hence V is finite dimensional.

Additional Problem. First we explain how to view $L(U_1, W)$ as a subspace of $L(V, W)$. Let $T : U_1 \to W$ be a linear transformation. Then there is a linear transformation $\tilde{T} : V \to W$ defined as follows: given any $v \in V$, we can (uniquely) write $v = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$. Then we set $\tilde{T}(u_1 + u_2) = T(u_1)$. We leave it to the reader to check that $\tilde{T}$ is a linear transformation (this is much easier to do yourself than to read a proof). This gives us a map $\iota_1 : L(U_1, W) \to L(V, W)$. In fact, ι_1 is a linear transformation, as I encourage you to check. Finally, ι_1 is injective: for if $\iota_1(T) = \tilde{T} = 0$, this means that $\tilde{T}(u_1 + u_2) = T(u_1) = 0$ for any $u_1 \in U_1$ and $u_2 \in U_2$, implying that $T = 0$. Therefore ι_1 induces an isomorphism (bijective linear map) from $L(U_1, W)$ onto its range in $L(V, W)$, which we denote by $\mathscr{L}(U_1, W)$. Similarly, we can define an injective linear map $\iota_2 : L(U_2, W) \to L(V, W)$ by $\iota_2(T)(u_1 + u_2) = T(u_2)$ for a given $T : U_2 \to W$. We let $\mathscr{L}(U_2, W)$ denote the range of ι_2. To summarize, viewing $L(U_1, W)$ and $L(U_2, W)$ as subspaces of $L(V, W)$ is the same as identifying $L(U_1, W)$ with $\mathscr{L}(U_1, W)$, and $L(U_2, W)$ with $\mathscr{L}(U_2, W)$.

Now we show that $L(V, W) = \mathscr{L}(U_1, W) \oplus \mathscr{L}(U_2, W)$. This is much easier to do by proving a lemma first:

Lemma 1. *$\mathscr{L}(U_1, W)$ consists of the linear transformations $T : V \to W$ such that $T(U_2) = 0$ (i.e., T restricts to the zero function on the subspace U_2). Similarly, $\mathscr{L}(U_2, W)$ consists of the linear transformations $T : V \to W$ such that $T(U_1) = 0$.*

Proof. It suffices to show the statement for U_1. From the definition of $\iota_1 : L(U_1, W) \to L(V, W)$, it is clear that for any $T \in \mathscr{L}(U_1, W)$ we have $T(U_2) = 0$, since U_2 consists of vectors of the form $0 + u_2$, for $u_2 \in U_2$. Conversely, suppose $T(U_2) = 0$. Define a linear transformation $\overline{T} : U_1 \to W$ to be the restriction of T to U_1. Then for any $v = u_1 + u_2 \in V$,

$$\iota_1(\overline{T})(u_1 + u_2) = \overline{T}(u_1) = T(u_1) = T(u_1) + T(u_2) = T(u_1 + u_2),$$

showing that $\iota_1(\overline{T}) = T$ and therefore T lies in the range of ι_1. This proves the lemma. □

We can now show quite easily that $\mathscr{L}(U_1, W) \cap \mathscr{L}(U_2, W) = \{0\}$, since according to the lemma this intersection consists of those $T : V \to W$ such that $T(U_1) = T(U_2) = 0$. But then for any $v = u_1 + u_2 \in V$, $T(u_1 + u_2) = T(u_1) + T(u_2) = 0$.

Finally, we show that $\mathscr{L}(U_1, W) + \mathscr{L}(U_2, W) = \mathscr{L}(V, W)$. Given any $T : V \to W$, define $\overline{T}_1 : U_1 \to W$ to be the restriction of T to the subspace U_1, i.e., $\overline{T}_1(u_1) = T(u_1)$. Similarly, define $\overline{T}_2 : U_2 \to W$ to be the restriction of T to U_2. Then we claim that

$$T = \iota_1(\overline{T}_1) + \iota_2(\overline{T}_2),$$

which will finish the proof. To show this, we apply both sides to an element $v = u_1 + u_2$ of V. On the left-hand side we have $T(u + v)$. To compute the right hand side, note that

$$\iota_1(\overline{T}_1)(u_1 + u_2) = \overline{T}_1(u_1) = T(u_1),$$

and similarly $\iota_2(\overline{T}_1)(u_1 + u_2) = T(u_2)$. Therefore the fact that $T = \iota_1(\overline{T}_1) + \iota_2(\overline{T}_2)$ follows from the fact that T is linear.