Homework 11 Solutions. Math 110, Fall 2013.

1. a) Suppose that T were self-adjoint. Then, the Spectral Theorem tells us that there would exist an orthonormal basis of $P_2(\mathbb{R})$, (p_1, p_2, p_3), consisting of eigenvectors of T. It is straightforward to see, by inspection of T, that the eigenvalues of T are $\lambda = 0, 1$ and that

$$T(v) = 0 \iff v \in \text{null}(T) = \{a_0 + a_2 x^t \mid a_0, a_2 \in R\},$$

$$T(v) = v \iff v \in \text{span}(x).$$

Thus, we have $p_1, p_2 \in \text{null}(T)$, with $\langle p_1, p_2 \rangle = 0$ and $||p_1|| = ||p_2|| = 1$. The remaining eigenvector p_3 must be an element of $\text{span}(x)$, so that $p_3 = cx$, for some (nonzero) $c \in \mathbb{R}$ such that $||p_3|| = 1$. Moreover, we would require that $\text{null}(T) \subset \text{span}(x)^\perp$, since eigenvectors associated to distinct eigenvalues are orthogonal. Therefore, $\text{null}(T) = \text{span}(x)^\perp$, by dimension considerations ($\dim \text{span}(x)^\perp = \dim P_2 - \dim \text{span}(x) = 3 - 1 - 2$).

However, if $p = a_0 + a_2 x^2 \in \text{span}(x)^\perp$ then

$$0 = \langle a_0 + a_2 x^2, x \rangle = \int_0^1 (a_0 + a_2 x^2) x \, dx = \frac{1}{4}(2a_0 + a_2) \implies a_2 = -2a_0$$

so that $p = a_0(1 - 2x^2)$. That is, we have shown that $\text{span}(1 - 2x^2) = \text{span}(x)^\perp \cap \text{null}(T) = \text{null}(T)$, since $\text{null}(T) = \text{span}(x)^\perp$. Then, we would have

$$2 = \dim \text{null}(T) = \dim \text{span}(1 - 2x^2) = 1$$

which is absurd. Hence, our assumption that T is self-adjoint is false.

b) Theorem 6.47 requires that the matrix of T^* (relative to $\mathcal{C} \subset W$ and $\mathcal{B} \subset V$) is the conjugate transpose of the matrix of T (relative to $\mathcal{B} \subset V$ and $\mathcal{C} \subset W$) if both $\mathcal{B}$ and $\mathcal{C}$ are orthonormal. However, the basis $\mathcal{B} = \mathcal{C} = (1, x, x^2)$ of P_2 is not orthonormal (with respect to the given inner product) so that we are not contradicting Theorem 6.47. If we chose an orthonormal basis of P_2, call it $\mathcal{A}$ (obtained by Gram-Schmidt process on $(1, x, x^2)$, for example), then we would find $[T]_\mathcal{A} \neq \overline{[T]}^t_\mathcal{A}$.

2. This is false. This would imply that the matrices A, B of two self-adjoint operators T, S (relative to an orthonormal basis) would satisfy

$$(AB)^* = AB,$$

where for a square matrix C we are writing $C^* = \overline{C}^t$. However, $(AB)^* = B^*A^* = BA$, since T, S are self-adjoint. So, we need to only find two non-commuting self adjoint operators - we can take the following operators on Euclidean space $\mathbb{C}^2$

$$T : \mathbb{C}^2 \to \mathbb{C}^2 \; ; \; \underline{x} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underline{x}, \quad S : \mathbb{C}^2 \to \mathbb{C}^2 \; ; \; \underline{x} \mapsto \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \underline{x}$$

You can check that $TS(e_1) \neq ST(e_1)$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3. a) Let $T, S \in L(V)$, be self-adjoint operators on V, a real inner product space. Then, the zero operator on V, $Z : V \to V \; ; \; v \mapsto 0_V$ is self-adjoint; we have $(T+S)^* = T^* + S^* = T + S$, so that $T + S$ is self-adjoint; if $c \in \mathbb{R}$ then $(cT)^* = \overline{c}T^* = cT$, so that cT is self-adjoint. Hence, the set of self-adjoint operators on a real inner product space is a subspacae of $L(V)$.

b) If T is self-adjoint operator on the complex inner product space V, then $(\sqrt{-1}T)^* = \overline{\sqrt{-1}}T^* = -\sqrt{-1}T \neq \sqrt{-1}T$. Hence, the set of self-adjoint operators is not closed under scalar multiplication.

4. Let $P \in L(V)$ be such that $P^2 = P$.

($\Rightarrow$) Suppose that P is an orthogonal projection. Then, $\text{range}(P) = \text{null}(P)^\perp$. Moreover, the only eigenvalues of P are $\lambda = 0, 1$ (this was proved in a previous HW exercise) and

$$P(v) = v \;\Leftrightarrow\; v \in \text{range}(P)$$

(this is true of any projection) so that range(P) consists of eigenvectors with eigenvalue $\lambda = 1$. Hence, we can find an orthonormal basis $(u_1, \dots, u_k)$of range(P) (using Gram-Schmidt applied to any basis of range(P)) and an orthonormal basis $(v_1, \dots, v_l)$ of null(P) (using Gram-Schmidt applied to any basis of null(P)). Then, $(u_1, \dots, u_k, v_1, \dots, v_l)$ is an orthonormal basis of V consisting of eigenvectors of P. Hence, if V is a real inner product space then P is self-adjoint, by the Spectral Theorem. If V is complex inner product space then, for any $v \in V$, we can write $v = u + z$, $u \in \text{range}(P), z \in \text{null}(P)$ so that

$$\langle P(v), v\rangle = \langle u, u + z\rangle = \langle u, u\rangle + \langle u, z\rangle = ||u||^2 + 0 \in \mathbb{R}$$

Hence, P is self-adjoint when V is complex inner product space.

($\Rightarrow$) Suppose that P is self-adjoint. Then, we have $\text{null}(P) = \text{null}(P^*$ and

$$\text{range}(P) = \text{null}(P^*)^\perp = \text{null}(P)^\perp \implies V = \text{null}(P) \oplus \text{null}(P)^\perp = \text{null}(P) \oplus \text{range}(P)$$

Since P is self-adjoint then there is a basis of V consisting of orthonormal vectors of P - call it $(u_1, \dots, u_k, v_1, \dots, v_l)$, where $P(u_i) = 0_V$ and $P(v_i) \neq 0_V$. Hence, $\dim \text{null}(P) = k$ (ie, we are saying that the u's are an o.n. basis of null(P)) and, since $\text{span}(v_1, \dots, v_l) \subset \text{null}(P)^\perp = \text{range}(P)$ and $\dim \text{range}(P) = \dim V - \dim \text{null}(P) = (k + l) - k = l$, we see that $\text{span}(v_1, \dots, v_l) = \text{range}(P)$. Thus, $(v_1, \dots, v_l)$ is an orthonormal basis of range(P) consisting of eigenvectors of P with nonzero associated eigenvalues. As we are assuming that $P^2 = P$, we must have that the only eigenvalues of P are $\lambda = 0, 1$, so that the only nonzero eigenvalue is $\lambda = 1$. Hence, for every $u \in \text{range}(P)$ we have $P(u) = u$. Thus, since we can write $v = z + u$, with $z \in \text{null}(P), u \in \text{range}(P)$, we see that

$$P(v) = P(z + u) = P(z) + P(u) = 0_V + u,$$

so that P is a projection onto range(P) with $\text{null}(P) = \text{range}(P)^\perp$ - hence, it is an orthogonal projection.

5. Let V be an inner product space, take $(v_1, \dots, v_n)$ an orthonormal basis of V (so that $n \geq 2$). Consider the normal operators $T, S \in L(V)$ defined as follows:

$$T(v_1) = 3v_1,\; T(v_2) = v_2,\; T(v_i) = 0_V,\; i \geq 3,$$

$$S(v_1) = v_2,\; S(v_2) = -v_1,\; S(v_i) = 0_V,\; i \geq 3.$$

Then, the $(n \times n)$ matrices of T, S with respect to the given basis are

$$A = \begin{bmatrix} 3 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Since, $A = \overline{A}^t$, we have $T = T^*$, and as $B\overline{B}^t = \overline{B}^t B$, we have $SS^* = S^*S$, giving that both T and S are normal.

Now, we see that the matrix of $T + S$ is

$$A + B = \begin{bmatrix} 3 & -1 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and this last matrix is not diagonalisable (so that $T + S$ is not diaagonalisable: indeed, the eigenvalues of $T + S$ are $\lambda = 0, 2$ and

$$\text{null}(T + S) = \text{span}(v_3, \dots, v_n)$$

while the $\lambda = 2$ eigenspace is $\text{span}(v_1 + v_2)$. If $T + S$ were to be diagonalisable then we would need to have two linearly independent eigenvectors with eigenvalue $\lambda = 2$, which obviously can't be the case. Hence, $T + S$ is not normal (it isn't diagonalisable).

6. Let $T \in L(V)$ be normal. Then, we must have that $\text{null}(T) = \text{null}(T^*)$ (this is at the top of p.131). Hence,

$$\text{range}(T) = \text{null}(T^*)^\perp = \text{null}(T)^\perp = \text{range}(T^*).$$

7. *There are a couple of ways to proceed:*

Proof I) Let B be an orthonormal basis of V consisting of eigenvectors of T (it exists by the Spectral Theorem). Then, we have the matrix of T relative to B is

$$[T]_B = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of T (counted with multiplicity). Let's suppose that $\lambda_1 = \dots = \lambda_k = 0$, and $\lambda_i \neq 0$, for $i > k$. Thus, $\dim \text{null}(T) = k$. Now, for any $j \geq 1$ we have

$$[T^j]_B = [T]_B^j = [T]_B = \begin{bmatrix} \lambda_1^j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^j \end{bmatrix}$$

and $\lambda_r^j = 0 \implies \lambda_r = 0 \implies r \in \{1, \dots, k\}$. Hence, $\text{null}(T^j) = \text{span}(v_1, \dots, v_k) = \text{null}(T)$, for each $j \geq 1$.

Now, since, for each $j \geq 1$,

$$\dim \text{range}(T) = \dim V - \dim \text{null}(T) = \dim V - \dim \text{null}(T^j) = \dim \text{range}(T^j)$$

and $\text{range}(T^j) \subset \text{range}(T)$, we see that $\text{range}(T) = \text{range}(T^j)$ follows from $\text{null}(T) = \text{null}(T^j)$.

Proof II) As T is normal then we have $\text{null}(T) = \text{null}(T^*)$ (see p.131). Hence, we have

$$\text{range}(T) = \text{null}(T^*)^\perp = \text{null}(T)^\perp \implies V = \text{null}(T) \oplus \text{range}(T)$$

In particular, $\text{null}(T) \cap \text{range}(T) = \{0\}$. Let's prove $\text{null}(T^j) = \text{null}(T)$, for every $j \geq 1$, by induction: the case $j = 1$ is trivial. Assume the result hold for $j = s$ - we'll show it holds for $j = s + 1$. Since $\text{null}(T) \subset \text{null}(T^{s+1})$ always holds, we need only show that $\text{null}(T) \supset \text{null}(T^{s+1})$. So, let $z \in \text{null}(T^{s+1})$. Then,

$$0 = T^{s+1}(z) = T(T^s(z)) \implies T^s(z) \in \text{null}(T) \cap \text{range}(T) = \{0\}$$

$$\implies z \in \text{null}(T^s) = \text{null}(T), \text{ by induction.}$$

Hence, $\text{null}(T^{s+1}) \subset \text{null}(T)$ and the result is proved.

8. The requirements on T imply that the vectors $(1, 2, 3)$ and $(2, 5, 7)$ are eigenvectors of T. However, with respect to the dot product on $\mathbb{R}^3$, we see that

$$(1, 2, 3) \cdot (2, 5, 7) = 2 + 10 + 21 = 33 \neq 0$$

so that eigenvectors corresponding to distinct eigenvalues are not orthogonal, contradicting Corollary 7.8.

9. ($\Rightarrow$) Suppose that T is self-adjoint. Then, by Proposition 7.1 we see that all eigenvalues of T are real.

($\Leftarrow$) Suppose that all eigenvalues of the normal operator T are real. Then, by the (complex) Spectral Theorem, we can find an orthonormal basis B of V consisting of eigenvectors of T. Hence, we have the matrix of T relative to B is

$$[T]_B = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \text{ where } \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

Then, we have that

$$[T^*]_B = \overline{[T]}_B^t = [T]_B \implies T = T^*.$$

Hence, T is self-adjoint.

10. Since T is normal, there is an orthonormal basis B of V consisting of eigenvectors of T. Then, we have

$$[T]_B = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \text{ and } [T^i]_B = \begin{bmatrix} \lambda_1^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^i \end{bmatrix}.$$

Hence, if $T^8 = T^9$ then we must have

$$[T^8]_B = \begin{bmatrix} \lambda_1^8 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^8 \end{bmatrix} = \begin{bmatrix} \lambda_1^9 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^9 \end{bmatrix} = [T^9]_B$$

so that, foe each $i = 1, \dots, n$

$$\lambda_i^8 = \lambda_i^9 \implies \lambda_i^8(1 - \lambda_i) = 0$$

In particular, each eigenvalue λ_i is either equal to 1 or 0. Since the eigenvalues of T are real then T is self-adjoint (by previous exercise). Moreover, if we assume that $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_{k+1} = \ldots = \lambda_n = 1$ then we have

$$[T]_B = \begin{bmatrix} \lambda_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = [T]_B$$

so that $T^2 = T$.

11. Let T be normal and $B = (v_1, \ldots, v_n)$ be an orthonormal basis of V consisting of eigenvectors of T. Suppose that

$$[T]_B = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Then, by the Fundamental Theorem of Algebra, we can find a (complex) square root of λ_i, for each $i = 1, \ldots, n$. Suppose that $\mu_i^2 = \lambda_i$, for each i. Then, define the operators $S \in L(V)$ as follows:

$$S(v_1) = \mu_1 v_1, \ \ldots \ , S(v_n) = \mu_n v_n$$

Then, we have

$$[S^2]_B = \begin{bmatrix} \mu_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = [T]_B \implies S^2 = T.$$