Solution to Midterm 2

1. Let S and T be the linear maps associated to the matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, respectively. Then ST is the map associated to the zero matrix, i.e., the zero map, whereas TS is equal to S, and hence not the zero map. In general, since the ranks of T and S are both at least that of TS, they must be nonzero. But neither can have rank two, or else it would be an isomorphism, and then $ST = 0$ would imply that the other was the zero map. Thus their ranks must both be one.

2. Letting $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{R}^3$, the map T sends e_1 to $(2, -1)$, e_2 to $(1, 0)$, and e_3 to $(0, 1)$. We need to find v_1, v_2, v_3 such that T sends v_1 to $(1, 0)$, v_2 to $(0, 1)$ and v_3 to zero. It's clear we should take $v_1 = e_2$ and $v_2 = e_3$. Since $T(e_1) = 2T(e_2) - T(e_3)$, we should take $v_3 = e_1 - 2e_2 + e_3$, so that $T(v_3)$ is zero. These three form an independent set since $\{v_1, v_2\}$ is independent, and v_3 is not in their span. There are other possible answers - but any choice of v_1, v_2, v_3 must differ from the above choices by a multiple of $(1, -2, 1)$.

3. The dual space V^* of a vector space V is the set of all linear functions from V to $\mathbb{F}$. If $\{v_1, \ldots, v_n\}$ is a basis for V, the dual basis $\{v_1^*, \ldots, v_n^*\}$ is defined by the condition that $v_i^*(v_j)$ is one if $i = j$, and zero otherwise. To find the matrix of T with respect to the basis β, we must compute $T(v_i)$ for each i and express it as a linear combination of the v_i. We have $T(v_i) = \sum v_j^*(v_i)v_j = v_i$. So the ith column of $[T]_\beta$ is just $(0, \ldots, 1, \ldots, 0)$, where the 1 is in the ith position; hence the matrix $[T]_\beta$ is the identity matrix.

4. (a) False - the equation $Ax = 0$ has only one solution if and only if left-multiplication by A is one-to-one. But if $m > n$ it will not be onto, in which case for some b, there will be no solution. For example, $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) True - if, as above, multiplication by A is one-to-one, then we have $n \leq m$; but if also also $m \leq n$, then $m = n$ and so A is invertible, so for any b there is a unique solution given by $x = A^{-1}b$.

5. (a) This is linear: for any $p(x), q(x) \in p_2(\mathbb{R})$, we have $f(p + q) = (p + q)(1) = p(1) + q(1) = f(p) + f(q)$ by definition of addition of polynomials. Similarly, if $c \in \mathbb{R}$, $f(cp) = (cp)(1) = c(p(1))$, by definition of scalar multiplication in $P_2(\mathbb{R})$, and this is equal to $cf(p)$.

This f is clearly onto, so its nullspace will be two-dimensional. Since $x - 1$ and $x^2 - 1$ are both in $N(f)$ and independent, they form a basis for $N(f)$.

(b) This is also linear - it is the composite of the map $p(x) \mapsto p(0)$ and the map $p(x) \mapsto p''(x)$, both of which are linear (the first one by the same argument as in (a), and the second by calculus). Alternatively, you could observe that since the second derivative of any quadratic is constant, this map is actually the same as the second derivative map! Again, it's clearly onto, since, e.g., $f(x^2) = 1$, and 1 is a basis for $\mathbb{R}$. So $N(f)$ is two dimensional. The polynomials 1 and x are both in $N(f)$ and are independent, so they form a basis for $N(f)$.

6. A and B are similar if there is an invertible matrix Q such that $B = Q^{-1}AQ$. Suppose this holds. Then

$$B^2 - B + I = (Q^{-1}AQ)^2 - Q^{-1}AQ + I = Q^{-1}AQQ^{-1}AQ - Q^{-1}AQ + Q^{-1}Q = Q^{-1}(A^2 - A + I)Q,$$

so $B^2 - B + I$ is similar to $A^2 - A + I$.