Solution to Midterm 1

1. The span of S is the same as the span of the rows of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

which is the same as the span of the rows of the row-reduced form

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus the set $\{(1,0,-1),(0,1,2)\}$ forms a basis for the span of S, which is therefore two-dimensional.

2. The set $\{v_4\}$ is independent since v_4 is nonzero. By the replacement theorem, there exists a subset H of S containing exactly two vectors such that $H \cup \{v_4\}$ generates $\mathbb{R}^3$. Neither element of H can be v_4, or else $H \cup \{v_4\} = H$ and this could not generate $\mathbb{R}^3$. Thus the subset $H \cup \{v_4\}$ of S consists of three vectors and spans $\mathbb{R}^3$, so it's a basis for $\mathbb{R}^3$.

Extra credit: The subset may not be unique: for example, for $i = 1,2,3$, let $v_i = e_i$, the standard basis for $\mathbb{R}^3$, and consider the following possible values of v_4. If $v_4 = 2e_1$, then there is only one subset containing v_4 which is a basis, namely $\{v_2, v_3, v_4\}$. If instead $v_4 = (0,1,1)$, then there are two possible subsets, whereas if $v_4 = (1,1,1)$, then there are three possible subsets.

3. A vector space V is infinite-dimensional if and only if there is no finite subset of V which generates V. Now let $V = P(\mathbb{R})$. Suppose for contradiction that V is *finite*-dimensional. Then there is a finite subset $p_1(x), \ldots, p_r(x)$ which generates V. Each of the polynomials $p_i(x)$ has a degree - let d denote the maximum of their degrees. Then any linear combination of the $p_i(x)$ has degree at most d, which contradicts the fact that they generate V, since there are polynomials in V with degree greater than d.

4. (a) is false: let W_2 be the line spanned by the vector $(1,1)$. Then $(0,1) \notin W_2$, but still $\mathbb{R}^2 = W_1 \oplus W_2$.

(b) is true: it's enough to verify that $S = \{(1,1,0),(0,1,1),(1,1,1)\}$ is a basis for $\mathbb{R}^3$. But the set $\{(1,1,0),(0,1,1)\}$ is clearly independent, and $(1,1,1)$ is not in its span since if $(1,1,1) = a(1,1,0) + b(0,1,1)$ then we must have $a = b = 1$, but $(1,1,1) \neq (1,1,0) + (0,1,1)$. Thus S is a basis for $\mathbb{R}^3$.

5. (a) This is a subspace. The polynomials $p_1(x) = 1, p_2(x) = x - x^2, p_3(x) = x - x^3$ are all in W, and are independent. They must be a basis for W; otherwise W would be four-dimensional and hence equal to $P_3(\mathbb{R})$, which is clearly not the case since, e.g. $x \notin W$. Hence W is three-dimensional.

(b) This is a subspace: the condition defining W is equivalent to $p(x) = p(1)x^3$; Such p must have the form $p(x) = ax^3$. Thus W is the one-dimensional subspace $\mathrm{Span}(x^3)$.

6. A field F has characteristic two if and only if $1 + 1 = 0$ in F. Given some nonzero $a \in F$ with $a = -a$, we have $2a = 0$, and since $a \neq 0$, a has a multiplicative inverse; multiplying through by this inverse gives $2 = 0$, so F has characteristic 2.