Solution to HW8

4.3.15 If A and B are similar, then $A = Q^{-1}BQ$ for some invertible Q. Taking determinants and using Thm 4.7 (and its corollary) gives

$$\det A = \det Q^{-1} \det B \det Q = \frac{1}{\det Q} \det B \det Q = \det B$$

4.3.21 We use induction on n. The base case[1] is when $n = 2$, in which case $M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for some $a, b, c \in \mathbb{F}$. Then $\det M = ac$, as desired.

Now suppose the result holds for matrices of the given form and of size $(n-1) \times (n-1)$, and let M be an $n \times n$ matrix of the given form. Performing a cofactor expansion along the first column gives

$$\det M = \sum_{i=1}^{n} (-1)^{i+1} M_{i1} \widetilde{M_{i1}}.$$

Letting A have size $k \times k$, then we have for $i > k$ that $M_{i1} = 0$ by the block-upper-triangular form of M. Thus

$$\det M = \sum_{i=1}^{k} (-1)^{i+1} M_{i1} \widetilde{M_{i1}}.$$

But for $i \leq k$, the matrix $\widetilde{M_{i1}}$ is of the form

$$\widetilde{M_{i1}} = \begin{pmatrix} \widetilde{A_{i1}} & \widetilde{B} \\ 0 & C \end{pmatrix},$$

where $\widetilde{B}$ is obtained from B by deleting the ith row (in fact, it doesn't actually matter what $\widetilde{B}$ is). Since this has size $(n-1) \times (n-1)$, the induction hypothesis applies, giving that $\det \widetilde{M_{i1}} = \det \widetilde{A_{i1}} \det C$. Putting this back in the formula above for $\det M$ gives

$$\det M = \sum_{i=1}^{k} (-1)^{i+1} \det \widetilde{A_{i1}} \det C = \left(\sum_{i=1}^{k} (-1)^{i+1} \det \widetilde{A_{i1}} \right) \det C = \det A \det C,$$

which completes the induction step. Thus the result holds for all $n \geq 2$.

[1] In case $n = 1$, the statement is vacuous, so we begin at $n = 2$.