Solutions to Homework #6.

Section 3.1.

12. We will prove the assertion by induction on the number of rows m.
For $m = 1$ there is nothing to prove.
Suppose we know the assertion for $m-1$ and would like to prove it for m. Let ℓ be the index of the first column of A that has a nonzero entry so that $A_{i,\ell'} = 0$ for any i and any $\ell' < \ell$. Let k be an index such that $A_{k,\ell} \neq 0$. By a row exchange, we may assume $k = 1$ so that $A_{1,\ell} \neq 0$.
Let B be the $(m-1)\times n$ matrix obtained from A by deleting its first row. Then by induction we can arrange using elementary row operations of types 1 and 3 that B is upper triangular. But then we see that the same operations applied to A with the first row undeleted make it upper triangular.

Section 3.2.

3. The rank of A is zero if and only if the span of the columns of A is zero if and only if all of the columns are zero or in other words A is the zero matrix.

4. We list sequences of matrices obtained by applying elementary operations.
(a)

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank = 2.
(b)

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 5/2 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Rank = 2.

5. (a)

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 5/2 & 1/2 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1/5 & 2/5 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 2/5 & -1/5 \\ 0 & 1 & 1/5 & 2/5 \end{array}\right)$$

Rank = 2, inverse above at right.
(b)

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

Rank = 1, not invertible.
(c)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right)$$

Rank = 2, not invertible.

7. Elementary operations:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Corresponding product of elementary matrices:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. (a) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f''(x) + 2f'(x) - f(x)$.
With respect to standard basis $\beta = \{1, x, x^2\}$, T takes the matrix form

$$T = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

It is invertible with inverse

$$T^{-1} = \begin{pmatrix} -1 & 2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$

(b) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = (x + 1)f'(x)$.
Observe that the null space $N(T)$ is non-zero, containing constant polynomials $p(x) = a$, so T is not invertible.

14. (a) $w \in R(T + U)$ means $w = (T + U)(v) = T(v) + U(v)$, for some $v \in V$. But $T(v) \in R(T)$ and $U(v) \in R(U)$ so $R(U + V) \subset R(U) + R(V)$.

(b) We've seen that for subspaces $W_1, W_2 \subset W$, if $\dim W$ is finite, then $\dim(W_1 + W_2) \leq \dim W_1 + \dim W_2$. Thus by part (a), we have $rk(T + U) = \dim R(T + U) \leq \dim(R(T) + R(U)) \leq \dim R(T) + \dim R(U) = rk(T) + rk(U)$.

(c) Rank and sum of linear transformations agrees with rank and sum of representing matrices, so this follows from part (b).

17. If $B \in M_{3\times 1}(F), C \in M_{1\times 3}(F)$, then $rk(BC) \leq rk(B) \leq 1$.

Conversely, suppose $A \in M_{3\times 3}(F)$ with $rk(A) = 1$. Let v be a nonzero vector in and hence a basis for the range $R(A)$. This provides an isomorphism $R(A) \simeq F$ such that v corresponds to 1. Thus we have factored A as a composition $F^3 \to F \to F^3$. Let $C \in M_{1\times 3}(F)$ be the first map and $B \in M_{3\times 1}(F)$ be the second.